

LOCAL STRUCTURE THEOREMS FOR SMOOTH MAPS OF FORMAL SCHEMES

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ABSTRACT. We deepen our study on infinitesimal lifting properties of maps between locally noetherian formal schemes started in [AJP]. In this paper, we focus on properties which make sense specifically in the formal context. In this vein, we make a detailed study of the relationship between the infinitesimal lifting properties of a morphism of formal schemes and those of the corresponding maps of usual schemes associated to the directed systems that define the corresponding formal schemes. Among our main results, we obtain the characterization of completion morphisms as pseudo closed immersions that are flat. Also, the local structure of smooth and étale morphisms between locally noetherian formal schemes is described: the former factors locally as a completion morphism followed by a smooth adic morphism and the latter as a completion morphism followed by an étale adic morphism.

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INTRODUCTION

Formal schemes have always been present in the backstage of algebraic geometry but they were rarely studied in a systematic way after the foundational [EGA I, §10]. It has become more and more clear that the wide applicability of formal schemes in several areas of mathematics require such study.

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Let us cite a few of these applications. The construction of De Rham cohomology for a scheme X of zero characteristic embeddable in a smooth scheme P , studied by Hartshorne [H] (and, independently, by Deligne), is defined as the hypercohomology of the formal completion of the De Rham complex of the completion of P along X . Formal schemes play a key role in p -adic cohomologies (crystalline, rigid ...) and are also algebraic models of rigid analytic spaces. These developments go back to Grothendieck with further elaborations by Raynaud in collaboration with Bosch and Lütkebohmert and later work by Berthelot and de Jong. In a different vein, Strickland [St] has pointed out the importance of formal schemes in the context of (stable) homotopy theory.

A particular assumption that it is almost always present in most earlier works on formal schemes is that morphisms are adic, *i.e.* that the topology of the sheaf of rings of the initial scheme is induced by the topology of the base formal scheme. This hypothesis on a morphism of formal schemes guarantees that its fibers are usual schemes, therefore an adic morphism between formal schemes is, in the terminology of Grothendieck's school, a relative scheme over a base that is a formal scheme. But there are important examples of maps of formal schemes that do not correspond to this situation. The first example that comes into mind is the natural map $\mathrm{Spf}(A[[X]]) \rightarrow \mathrm{Spf}(A)$ for an adic ring A . This morphism has a finiteness property that had not been made explicit until [AJL1] (and independently, in [Y]). This property is called *pseudo finite type*¹. The fact that pseudo finite type morphisms need not be adic allows fibers that are not usual schemes, and the structure of these maps is, therefore, more complex than the structure of adic maps. The study of smoothness and, more generally, infinitesimal lifting properties in the context of noetherian formal schemes together with this hypothesis of finiteness was embraced in general in our previous work [AJP]. We should mention a preceding study of smooth morphisms under the restriction that the base is a usual scheme in [Y] and also the overlap of several results in [AJP] and a set of results in [LNS, §2], based on Nayak's 1998 thesis.

In [AJP] we studied the good properties of these definitions and the agreement of their properties with the corresponding behavior for usual noetherian schemes, obtaining the corresponding statement of Zariski's Jacobian criterion for smoothness. Now we concentrate on studying properties which make sense specifically in the formal context getting information about the infinitesimal lifting properties from information present in the structure of a formal scheme.

This paper can be structured roughly into three parts. The first, formed by sections 1, 2 and 3 includes preliminaries, introduces the notion of quasi-covering and the study of completion morphisms. We know of no previous reference about these matters, so we include all the needed details. They will be indispensable to state our results. The second part encompasses

¹In [Y] the terminology *formally finite type* is used.

three sections (4, 5 and 6). We show that there exists a close relationship between the infinitesimal lifting properties of an adic morphism and the infinitesimal lifting properties of the underlying morphism of ordinary schemes f_0 . The third part (section 7) treats the structure theorems, which are the main results of this work. We characterize open immersions and completion morphisms in terms of the étale property. We classify étale adic coverings of a noetherian formal scheme. Finally, we give local structure theorems for unramified, étale and smooth maps, that show that it is possible to factor them locally into simpler maps.

Let us discuss in greater detail the contents of every section. Our framework is the category of locally noetherian formal schemes. In this category a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ can be expressed as a direct limit

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

of a family of maps of ordinary schemes using appropriate Ideals of definition. The first section sets the basic notations and recalls some definitions that will be used throughout the paper. The second section deals with morphisms between locally noetherian formal schemes expressed as before as a limit in which every map f_n is a closed immersion of usual schemes. It is a true closed immersion of formal schemes when f is adic. We treat radical maps of formal schemes and see that the main results are completely similar to the case of usual schemes. On usual schemes, quasi-finite maps play a very important role in the understanding of the structure of étale maps. In the context of formal schemes there are two natural generalizations of this notion. The simplest one is *pseudo-quasifinite* (Definition 2.7) — in a few words: “of pseudo-finite type with finite fibers”. The key notion though is that of quasi-covering (Definition 2.8). While both are equivalent in the context of usual schemes, the latter is a basic property of unramified and, therefore, étale maps between formal schemes (*cf.* Corollaries 4.7 and 6.6). In section 3 we discuss flat morphisms in the context of locally noetherian formal schemes. Next, we study morphisms of completion in this setting. They form a class of flat morphisms that are closed immersions as topological maps. These kind of maps will be essential for the results of the last section.

Expressing a morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ between locally noetherian formal schemes as a limit as before, it is sensible to ask about the relation that exists between the infinitesimal lifting properties of f and the infinitesimal lifting properties of the underlying morphisms of usual schemes $\{f_n\}_{n \in \mathbb{N}}$. This is one of the main topics of the next three sections. The case of unramified morphisms is simple: f is unramified if, and only if, f_n are unramified $\forall n \in \mathbb{N}$ (Proposition 4.1). Another characterization is that f is unramified if, and only if, f_0 is *and* the fibers of f and of f_0 agree (Corollary 4.10). A consequence of this result is a useful characterization of pseudo closed immersion as those unramified morphisms such that f_0 is a closed immersion

(Corollary 4.13). Smooth morphisms are somewhat more difficult to characterize. An *adic* morphism f is smooth if, and only if, f_0 is and f is flat (Corollary 5.6). For a non adic morphism, one cannot expect that the maps f_n are going to be smooth as it is shown by example 5.7. On the positive side, there is a nice characterization of smooth closed subschemes (Proposition 5.11). Also, the matrix jacobian criterion holds for formal schemes, see Corollary 5.13 for a precise statement. In section 6 we combine these results to obtain properties of étale morphisms. It is noteworthy to point out that a smooth pseudo quasi-finite map need not be étale (Example 6.7).

The last section contains our main results. First we recover in our framework the classical fact for usual schemes [EGA IV₄, (17.9.1)] that an open immersion is a map that is étale and radical (Theorem 7.3). We also characterize completion morphisms as those pseudo closed immersions that are flat. This and other characterizations are given in Proposition 7.5. Writing a locally noetherian formal scheme \mathfrak{Y} as

$$\mathfrak{Y} = \varinjlim_{n \in \mathbb{N}} Y_n$$

with respect to an Ideal of definition, Proposition 7.7 says that there is an equivalence of categories between étale adic \mathfrak{Y} -formal schemes and étale Y_0 -schemes. A special case already appears in [Y, Proposition 2.4]. In fact, this result is a reinterpretation of [EGA IV₄, (18.1.2)]. The factorization theorems are based in Theorem 7.11 that says that an unramified morphism can be factored locally into a pseudo closed immersion followed by an étale adic map. As consequences we obtain Theorem 7.12 and Theorem 7.13. They state that every smooth morphism and every étale morphism factor locally as a completion morphism followed by a smooth adic morphism and an étale adic morphism, respectively. These results explain the local structure of smooth and étale morphisms of formal schemes. It has been remarked by Lipman, Nayak and Sastry in [LNS, p. 132] that this observation may simplify some developments related to Cousin complexes and duality on formal schemes.

1. PRELIMINARIES

We denote by **NFS** the category of the locally noetherian formal schemes and by **NFS_{af}** the subcategory of locally noetherian affine formal schemes. We write **Sch** for the category of ordinary schemes.

We assume that the reader is familiar with the basic theory of formal schemes as is explained in [EGA I, §10]: formal spectrum, Ideal of definition of a formal scheme, fiber product of formal schemes, functor $M \rightsquigarrow M^\Delta$ for modules over adic rings, completion of a usual scheme along a closed subscheme, adic morphisms, separated morphisms, etc.

From now on and, except otherwise indicated, every formal scheme will be in **NFS**. We will assume that every ring is noetherian and, therefore,

that every complete ring and every complete module for an adic topology are separated.

1.1. Henceforward, the following notation [EGA I, §10.6] will be used:

- (1) Given $\mathfrak{X} \in \text{NFS}$ and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition for each $n \in \mathbb{N}$ we put $X_n := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ and we indicate that \mathfrak{X} is the direct limit of the schemes X_n by

$$\mathfrak{X} = \varinjlim_{n \in \mathbb{N}} X_n.$$

The ringed spaces \mathfrak{X} and X_n have the same underlying topological space, so we will not distinguish between a point in \mathfrak{X} or X_n .

- (2) If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is in NFS, $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ are Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and $f_n: X_n := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1}) \rightarrow Y_n := (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^{n+1})$ is the morphism induced by f , for each $n \in \mathbb{N}$, then f is expressed as

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

- (3) Furthermore, given $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition, there exist $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$.

1.2. Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. The morphism f is of *pseudo finite type* (*pseudo finite*) [AJL1, p.7] if f_0 (and in fact any f_n) is of finite type (finite). Moreover, if f is adic we say that f is of *finite type (finite)* [EGA I, 10.13.3] ([EGA III₁, (4.8.2)]).

1.3. [AJP, Definition 2.1 and Definition 2.6] A morphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS is *smooth (unramified, étale)* if it is of pseudo finite type and satisfies the following lifting condition:

For all affine \mathfrak{Y} -schemes Z and for each closed subscheme $T \hookrightarrow Z$ given by a square zero Ideal $\mathcal{I} \subset \mathcal{O}_Z$ the induced map

$$(1.3.1) \quad \text{Hom}_{\mathfrak{Y}}(Z, \mathfrak{X}) \longrightarrow \text{Hom}_{\mathfrak{Y}}(T, \mathfrak{X})$$

is surjective (injective, bijective).

Moreover, if f is adic we say that f is *smooth adic (unramified adic, étale adic)*.

1.4. (cf. [AJP, §3]) Given $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS the *differential pair of \mathfrak{X} over \mathfrak{Y}* , $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1, \widehat{d}_{\mathfrak{X}/\mathfrak{Y}})$, is locally given by $(\widehat{\Omega}_{A/B}^1, \widehat{d}_{A/B})$ for all open sets $\mathfrak{U} = \text{Spf}(A) \subset \mathfrak{X}$ and $\mathfrak{V} = \text{Spf}(B) \subset \mathfrak{Y}$ with $f(\mathfrak{U}) \subset \mathfrak{V}$. The $\mathcal{O}_{\mathfrak{X}}$ -Module $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1$ is called *module of 1-differentials of \mathfrak{X} over \mathfrak{Y}* and the continuous \mathfrak{Y} -derivation $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$ is called *canonical derivation of \mathfrak{X} over \mathfrak{Y}* .

1.5. [EGA I, p. 442] A morphism $f: \mathfrak{Z} \rightarrow \mathfrak{X}$ in NFS is a *closed immersion* if it factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{X}' \xrightarrow{j} \mathfrak{X}$ where g is an isomorphism of \mathfrak{Z} into a closed

subscheme $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ of the formal scheme \mathfrak{X} ([EGA I, (10.14.2)]). Let us recall [EGA III₁, (4.8.10)]: a morphism $f: \mathfrak{Z} \rightarrow \mathfrak{X}$ in NFS is a closed immersion if it is adic and given $\mathcal{K} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition of \mathfrak{X} and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{Z}}$, the corresponding Ideal of definition of \mathfrak{Z} the induced morphism $f_0: Z_0 \rightarrow X_0$ is a closed immersion, equivalently, the induced morphisms $f_n: Z_n \rightarrow X_n$ are closed immersions for all $n \in \mathbb{N}$.

A morphism $f: \mathfrak{Z} \rightarrow \mathfrak{X}$ in NFS is an *open immersion* if it factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{X}' \hookrightarrow \mathfrak{X}$ where g is an isomorphism of \mathfrak{Z} into an open subscheme $\mathfrak{X}' \hookrightarrow \mathfrak{X}$.

Definition 1.6. Let \mathfrak{X} be in NFS, $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition and $x \in \mathfrak{X}$. We define the *topological dimension of \mathfrak{X} at x* as

$$\dim_{\text{top}_x} \mathfrak{X} = \dim_x X_0.$$

It is easy to see that the definition does not depend on the chosen Ideal of definition of \mathfrak{X} . We define the *topological dimension of \mathfrak{X}* as

$$\dim_{\text{top}} \mathfrak{X} = \sup_{x \in \mathfrak{X}} \dim_{\text{top}_x} \mathfrak{X} = \sup_{x \in \mathfrak{X}} \dim_x X_0 = \dim X_0.$$

Given A an I -adic noetherian ring, put $X = \text{Spec}(A)$ and $\mathfrak{X} = \text{Spf}(A)$, then $\dim_{\text{top}} \mathfrak{X} = \dim A/I$. Despite the only “visible part” of \mathfrak{X} in $X = \text{Spec}(A)$ is $V(I)$, it happens that $X \setminus V(I)$ has a deep effect on the behavior of \mathfrak{X} as we will see along this work. So apart from the topological dimension of \mathfrak{X} , it is necessary to consider another notion of dimension that expresses part of the “hidden” information: the algebraic dimension.

Definition 1.7. Let \mathfrak{X} be in NFS and \mathcal{J} an Ideal of definition of \mathfrak{X} . Given $x \in \mathfrak{X}$ we define the *algebraic dimension of \mathfrak{X} at x* as

$$\dim_x \mathfrak{X} = \dim \mathcal{O}_{\mathfrak{X},x}.$$

The *algebraic dimension of \mathfrak{X}* is

$$\dim \mathfrak{X} = \sup_{x \in \mathfrak{X}} \dim_x \mathfrak{X}.$$

Proposition 1.8. *If $\mathfrak{X} = \text{Spf}(A)$ with A an I -adic noetherian ring then $\dim \mathfrak{X} = \dim A$.*

Proof. For each $x \in \mathfrak{X}$, if \mathfrak{p}_x is the corresponding open prime ideal in A we have that $\dim_x \mathfrak{X} = \dim A_{\{\mathfrak{p}_x\}} = \dim A_{\mathfrak{p}_x}$ since $A_{\mathfrak{p}_x} \hookrightarrow A_{\{\mathfrak{p}_x\}}$ is a flat extension of local rings with the same residue field (cf. [M1, (24.D)]). \square

Example 1.9. Given A an I -adic noetherian ring and $\mathbf{T} = T_1, T_2, \dots, T_r$ a finite number of indeterminates, the *affine formal space of dimension r over A* is $\mathbb{A}_{\text{Spf}(A)}^r = \text{Spf}(A\{\mathbf{T}\})$ and the *formal disc of dimension r over A* is $\mathbb{D}_{\text{Spf}(A)}^r = \text{Spf}(A[[\mathbf{T}]])$ (see [AJP, Example 1.6]). It holds that

$$\begin{aligned} \dim_{\text{top}} \mathbb{A}_{\text{Spf}(A)}^r &= \dim \mathbb{A}_{\text{Spec}(A/I)}^r = \dim A/I + r \\ \dim_{\text{top}} \mathbb{D}_{\text{Spf}(A)}^r &= \dim \text{Spec}(A/I) = \dim A/I \end{aligned}$$

and

$$\begin{aligned} \dim \mathbb{A}_{\mathrm{Spf}(A)}^r &\stackrel{1.8}{=} \dim A\{\mathbf{T}\} = \dim A + r \stackrel{1.8}{=} \dim \mathrm{Spf}(A) + r \\ \dim \mathbb{D}_{\mathrm{Spf}(A)}^r &\stackrel{1.8}{=} \dim A[[\mathbf{T}]] = \dim A + r \stackrel{1.8}{=} \dim \mathrm{Spf}(A) + r. \end{aligned}$$

From this examples, we see that the algebraic dimension of a formal scheme does not measure the dimension of the underlying topological space. In general, for \mathfrak{X} in NFS, $\dim_x \mathfrak{X} \geq \dim_{\mathrm{top}_x} \mathfrak{X}$, for any $x \in \mathfrak{X}$ and, therefore

$$\dim \mathfrak{X} \geq \dim_{\mathrm{top}} \mathfrak{X}.$$

Moreover, if $\mathfrak{X} = \mathrm{Spf}(A)$ with A an I -adic ring then $\dim \mathfrak{X} \geq \dim_{\mathrm{top}} \mathfrak{X} + \mathrm{ht}(I)$.

Definition 1.10. Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be in NFS and $y \in \mathfrak{Y}$. The *fiber of f at the point y* is the formal scheme

$$f^{-1}(y) = \mathfrak{X} \times_{\mathfrak{Y}} \mathrm{Spec}(k(y)).$$

For example, if $f : \mathfrak{X} = \mathrm{Spf}(B) \rightarrow \mathfrak{Y} = \mathrm{Spf}(A)$ is in NFS_{af} we have that $f^{-1}(y) = \mathrm{Spf}(B \widehat{\otimes}_A k(y))$.

Example 1.11. Let $\mathfrak{Y} = \mathrm{Spf}(A)$ be in NFS_{af} and $\mathbf{T} = T_1, T_2, \dots, T_r$. If $p : \mathbb{A}_{\mathfrak{Y}}^r \rightarrow \mathfrak{Y}$ is the canonical projection of the affine formal r -space over \mathfrak{Y} , for all $x \in \mathbb{A}_{\mathfrak{Y}}^r$ and $y = p(x)$ we have that

$$p^{-1}(y) = \mathrm{Spf}(A\{\mathbf{T}\} \widehat{\otimes}_A k(y)) = \mathrm{Spec}(k(y)[\mathbf{T}]) = \mathbb{A}_{\mathrm{Spec}(k(y))}^r.$$

If $q : \mathbb{D}_{\mathfrak{Y}}^r \rightarrow \mathfrak{Y}$ is the canonical projection of the formal r -disc over \mathfrak{Y} , given $x \in \mathbb{D}_{\mathfrak{Y}}^r$ and $y = q(x)$, there results that

$$q^{-1}(y) = \mathrm{Spf}(A[[\mathbf{T}]] \widehat{\otimes}_A k(y)) = \mathrm{Spf}(k(y)[[\mathbf{T}]]) = \mathbb{D}_{\mathrm{Spec}(k(y))}^r.$$

1.12. Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be in NFS and let us consider $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. According to 1.1, the morphism f can be written as $f = \varinjlim_{n \in \mathbb{N}} (f_n : X_n \rightarrow Y_n)$ with respect to the Ideals of

definition \mathcal{J} and \mathcal{K} . Then, by [EGA I, (10.7.4)] it holds that

$$f^{-1}(y) = \varinjlim_{n \in \mathbb{N}} f_n^{-1}(y)$$

where $f_n^{-1}(y) = X_n \times_{Y_n} \mathrm{Spec}(k(y))$, for each $n \in \mathbb{N}$.

If f is adic, by base-change (*cf.* [AJP, 1.3]) we deduce that $f^{-1}(y) \rightarrow \mathrm{Spec}(k(y))$ is adic so, $f^{-1}(y)$ is a (ordinary) scheme and $f^{-1}(y) = f_n^{-1}(y)$, for all $n \in \mathbb{N}$.

1.13. We will establish the following convention. Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be in NFS, $x \in \mathfrak{X}$ and $y = f(x)$ and assume that $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ are Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. From now and, except otherwise indicated, whenever we consider the rings $\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{O}_{\mathfrak{Y},y}$ we will associate them the $\mathcal{J}\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{K}\mathcal{O}_{\mathfrak{Y},y}$ -adic topologies, respectively. And we will denote by

$\widehat{\mathcal{O}}_{\mathfrak{X},x}$ and $\widehat{\mathcal{O}}_{\mathfrak{Y},y}$ the completion of $\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{O}_{\mathfrak{Y},y}$ with respect to the $\mathcal{J}\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{K}\mathcal{O}_{\mathfrak{Y},y}$ -adic topologies, respectively.

Definition 1.14. Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be in NFS. Given $x \in \mathfrak{X}$ and $y = f(x)$, we define the *relative algebraic dimension of f at x* as

$$\dim_x f = \dim_x f^{-1}(y)$$

If $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ are Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, then

$$\dim_x f = \dim \mathcal{O}_{f^{-1}(y),x} = \dim \mathcal{O}_{\mathfrak{X},x} \otimes_{\mathcal{O}_{\mathfrak{Y},y}} k(y) = \dim \widehat{\mathcal{O}}_{\mathfrak{X},x} \otimes_{\widehat{\mathcal{O}}_{\mathfrak{Y},y}} k(y).$$

The topology in $\widehat{\mathcal{O}}_{\mathfrak{X},x} \otimes_{\widehat{\mathcal{O}}_{\mathfrak{Y},y}} k(y)$ is the $\mathcal{J}\widehat{\mathcal{O}}_{\mathfrak{X},x}$ -adic then $\widehat{\mathcal{O}}_{\mathfrak{X},x} \widehat{\otimes}_{\widehat{\mathcal{O}}_{\mathfrak{Y},y}} k(y) = \widehat{\mathcal{O}}_{\mathfrak{X},x} \otimes_{\widehat{\mathcal{O}}_{\mathfrak{Y},y}} k(y)$.

1.15. Given an adic morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS and Ideals of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, then $\dim_x f = \dim_x f_0$ for every $x \in \mathfrak{X}$. For example:

- (1) If $p : \mathbb{A}_{\mathfrak{Y}}^r := \mathbb{A}_{\mathbb{Z}}^r \times_{\mathbb{Z}} \mathfrak{Y} \rightarrow \mathfrak{Y}$ is the canonical projection of the affine formal r -space over \mathfrak{Y} , given $x \in \mathbb{A}_{\mathfrak{Y}}^r$ we have that

$$\dim_x p = \dim k(y)[\mathbf{T}] = r,$$

where $y = p(x)$. In contrast, if $q : \mathbb{D}_{\mathfrak{Y}}^r := \mathbb{D}_{\mathbb{Z}}^r \times_{\mathbb{Z}} \mathfrak{Y} \rightarrow \mathfrak{Y}$ is the canonical projection of the formal r -disc over \mathfrak{Y} , $x \in \mathbb{D}_{\mathfrak{Y}}^r$ and $y = q(x)$ there results that

$$\dim_x q = \dim k(y)[[\mathbf{T}]] \stackrel{1.9}{=} r > \dim k(y) = 0$$

- (2) If X is a usual noetherian scheme and X' is a closed subscheme of X , recall that the morphism of completion of X along X' , $\kappa : X_{/X'} \rightarrow X$ ([EGA I, (10.8.5)]) is not adic, in general. Note however that

$$\dim_x \kappa = \dim k(x) = 0$$

for all $x \in X_{/X'}$.

2. PSEUDO CLOSED IMMERSIONS AND QUASI-COVERINGS

Definition 2.1. A morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS is a *pseudo closed immersion* if there exists $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition satisfying $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and such that the induced morphisms of usual schemes $\{f_n : X_n \rightarrow Y_n\}_{n \in \mathbb{N}}$ are closed immersions.

Note that if $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a pseudo closed immersion, $f(\mathfrak{X})$ is a closed subset of \mathfrak{Y} .

Let us show that this definition does not depend on the chosen Ideals of definition. Being a local question, we can assume that $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathrm{NFS}_{\mathrm{af}}$ and that $\mathcal{J} = J^{\Delta}$, $\mathcal{K} = K^{\Delta}$ for ideals of definition $J \subset A$ and $K \subset B$ such that $KA \subset J$. Then, given another pair of ideals of definition $J' \subset A$ and $K' \subset B$ such that $\mathcal{J}' = J'^{\Delta} \subset \mathcal{O}_{\mathfrak{X}}$, $\mathcal{K}' = K'^{\Delta} \subset \mathcal{O}_{\mathfrak{Y}}$

satisfying that $f^*(\mathcal{K}')\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}'$, there exists $n_0 > 0$ such that $J^{n_0} \subset J'$ and $K^{n_0} \subset K'$. The morphism $B \rightarrow A$ induces the following commutative diagrams

$$\begin{array}{ccc} B/K^{n_0(n+1)} & \longrightarrow & A/J^{n_0(n+1)} \\ \downarrow & & \downarrow \\ B/K^{m+1} & \longrightarrow & A/J^{m+1} \end{array}$$

and it follows that $B/K^{m+1} \rightarrow A/J^{m+1}$ is surjective, for all $n \in \mathbb{N}$. Then, using 1.5, it follows that the morphism $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{m+1}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^{m+1})$ is a closed immersion, for all $n \in \mathbb{N}$.

Example 2.2. Let X be a noetherian scheme and $X' \subset X$ a closed subscheme defined by an Ideal $\mathcal{I} \subset \mathcal{O}_X$. The morphism of completion $X_{/X'} \xrightarrow{\kappa} X$ of X along X' ([EGA I, (10.8.5)]) is expressed as

$$\varinjlim_{n \in \mathbb{N}} \left((X', \mathcal{O}_X/\mathcal{I}^{n+1}) \xrightarrow{\kappa_n} (X, \mathcal{O}_X) \right),$$

therefore, it is a pseudo closed immersion.

Notice that an adic pseudo closed immersion is a closed immersion (cf. 1.5). However, to be a pseudo closed immersion is not a topological property:

Example 2.3. Given K a field, let $p : \mathbb{D}_{\text{Spec}(K)}^1 \rightarrow \text{Spec}(K)$ be the canonical projection. If we consider the Ideal of definition $\langle T \rangle^\Delta$, of $\mathbb{D}_{\text{Spec}(K)}^1$ then $p_0 = 1_{\text{Spec}(K)}$ is a closed immersion. However, the morphisms

$$p_n : \text{Spec}(K[T]/\langle T \rangle^{n+1}) \rightarrow \text{Spec}(K)$$

are not closed immersions, for all $n > 0$ and, thus, p is not a pseudo closed immersion.

Proposition 2.4. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ be two morphisms in NFS. It holds that:*

- (1) *If f and g are (pseudo) closed immersions then $g \circ f$ is a (pseudo) closed immersion.*
- (2) *If f is a (pseudo) closed immersion, given $h : \mathfrak{Y}' \rightarrow \mathfrak{Y}$ in NFS we have that $\mathfrak{X}_{\mathfrak{Y}'} = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ is in NFS and that $f' : \mathfrak{X}_{\mathfrak{Y}'} \rightarrow \mathfrak{Y}'$ is a (pseudo) closed immersion.*

Proof. It is known that adic morphisms are stable under composition and base-change (cf. [AJP, 1.3]) so it suffices to show the properties for pseudo closed immersions. As for (1) let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$, $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{L} \subset \mathcal{O}_{\mathfrak{S}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, $g^*(\mathcal{L})\mathcal{O}_{\mathfrak{Y}} \subset \mathcal{K}$ and consider the corresponding expressions for f and g as direct limit of scheme morphisms:

$$f = \varinjlim_{n \in \mathbb{N}} (X_n \xrightarrow{f_n} Y_n) \quad g = \varinjlim_{n \in \mathbb{N}} (Y_n \xrightarrow{g_n} S_n)$$

Since

$$g \circ f = \varinjlim_{n \in \mathbb{N}} g_n \circ f_n$$

the assertion follows from the stability under composition of closed immersions in Sch . Let us show (2). Take $\mathcal{K}' \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition with $h^*(\mathcal{K})\mathcal{O}_{\mathfrak{Y}} \subset \mathcal{K}'$ and such that, by 1.1,

$$h = \varinjlim_{n \in \mathbb{N}} (h_n : Y'_n \rightarrow Y_n).$$

Then by [EGA I, (10.7.4)] we have that

$$\begin{array}{ccc} \mathfrak{X}_{\mathfrak{Y}} & \xrightarrow{f'} & \mathfrak{Y}' \\ \downarrow & & \downarrow h \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array} = \varinjlim_{n \in \mathbb{N}} \left(\begin{array}{ccc} X_n \times_{Y_n} Y'_n & \xrightarrow{f'_n} & Y'_n \\ \downarrow & & \downarrow h_n \\ X_n & \xrightarrow{f_n} & Y_n \end{array} \right)$$

By hypothesis, f_n is a closed immersion and since closed immersions in Sch are stable under base-change we have that f'_n is a closed immersion of noetherian schemes, $\forall n \in \mathbb{N}$. Finally, since f is a morphism of pseudo finite type, from [AJP, Proposition 1.8.(2)] we have that $\mathfrak{X}_{\mathfrak{Y}}$ is in NFS. \square

Next we turn to the study of radical morphisms in the context of formal schemes. This notion will allow us later (Theorem 7.3) to give a characterization of open immersions in terms of étale morphisms.

Definition 2.5. A morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS is *radical* if given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ the induced morphism of schemes $f_0 : X_0 \rightarrow Y_0$ is radical.

Given $x \in \mathfrak{X}$, the residue fields of the local rings $\mathcal{O}_{\mathfrak{X},x}$ and $\mathcal{O}_{X_0,x}$ agree and analogously for $\mathcal{O}_{\mathfrak{Y},f(x)}$ and $\mathcal{O}_{Y_0,f_0(x)}$. Therefore the definition of radical morphisms does not depend on the chosen Ideals of definition of \mathfrak{X} and \mathfrak{Y} .

2.6. From the sorites of radical morphisms in Sch it follows that:

- (1) Radical morphisms are stable under composition and noetherian base-change.
- (2) Every monomorphism is radical. So, open immersions, closed immersions and pseudo closed immersions are radical morphisms.

The notion of quasi-finite morphism of usual schemes [EGA I, Definition(6.11.3)] is based on the equivalence between several conditions for morphisms between schemes (see Corollaire (6.11.2) in *loc. cit.*) that are no longer equivalent in the full context of formal schemes. Specifically, we study two notions that generalize that of quasi-finite morphism of usual schemes. They will play a basic role in understanding the structure of unramified and étale morphisms in NFS.

Definition 2.7. Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ a pseudo finite type morphism in NFS. We say that f is *pseudo quasi-finite* if there exist $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and such that f_0 is quasi-finite. And f is *pseudo quasi-finite at $x \in \mathfrak{X}$* if there exists an open neighborhood $x \in \mathfrak{U} \subset \mathfrak{X}$ such that $f|_{\mathfrak{U}}$ is pseudo quasi-finite.

Notice that if $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a pseudo quasi-finite morphism (in NFS) then, for all couples of Ideals of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, the induced morphism of schemes $f_0: X_0 \rightarrow Y_0$ is quasi-finite.

As an immediate consequence of the analogous properties in Sch we have that:

- (1) Closed immersions, pseudo closed immersions and open immersions are pseudo quasi-finite.
- (2) Pseudo finite morphisms and finite morphisms are pseudo quasi-finite.
- (3) If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{S}$ are pseudo quasi-finite morphisms, then $g \circ f$ also is.
- (4) If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is pseudo quasi-finite, given $h: \mathfrak{Y}' \rightarrow \mathfrak{Y}$ a morphism in NFS we have that $f': \mathfrak{X}_{\mathfrak{Y}'} \rightarrow \mathfrak{Y}'$ is pseudo quasi-finite.

In Sch it is the case that a morphism is étale if, and only if, it is smooth and quasi-finite. Nevertheless, we will show that in NFS not every smooth and pseudo quasi-finite morphism is étale (see Example 6.7). That is why we introduce a stronger notion than pseudo quasi-finite morphism and that also generalizes quasi-finite morphisms in Sch: the quasi-coverings.

Definition 2.8. Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. The morphism f is a *quasi-covering* if $\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},f(x)}} k(f(x))$ is a finite type $k(f(x))$ -module, for all $x \in \mathfrak{X}$. We say that f is a *quasi-covering at $x \in \mathfrak{X}$* if there exists an open $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ is a quasi-covering.

We reserve the word *covering* for a dominant (*i.e.* with dense image) quasi-covering. These kind of maps will play no role in the present work but they are important, for instance, in the study of finite group actions on formal schemes.

Example 2.9. If X is a locally noetherian scheme and $X' \subset X$ is a closed subscheme the morphism of completion $\kappa: \mathfrak{X} = X_{/X'} \rightarrow X$ is a quasi-covering. In fact, for all $x \in \mathfrak{X}$ we have that

$$\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{X,\kappa(x)}} k(\kappa(x)) = k(\kappa(x)).$$

Lemma 2.10. *We have the following:*

- (1) *Closed immersions, pseudo closed immersions and open immersions are quasi-coverings.*
- (2) *If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g: \mathfrak{Y} \rightarrow \mathfrak{S}$ are quasi-coverings, the morphism $g \circ f$ is a quasi-covering.*
- (3) *If $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ is a quasi-covering, and $h: \mathfrak{Y}' \rightarrow \mathfrak{Y}$ a morphism in NFS, then $f': \mathfrak{X}_{\mathfrak{Y}'} \rightarrow \mathfrak{Y}'$ is a quasi-covering.*

Proof. Immediate. □

Proposition 2.11. *If $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a quasi-covering in $x \in \mathfrak{X}$ then:*

$$\dim_x f = 0$$

Proof. It is a consequence of the fact that $\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},f(x)}} k(f(x))$ is an artinian ring. □

Remark. Observe that given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{K}$, for all $x \in \mathfrak{X}$ there results that

$$\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},f(x)}} k(f(x)) = \varprojlim_{n \in \mathbb{N}} \mathcal{O}_{X_n,x} \otimes_{\mathcal{O}_{Y_n,f_n(x)}} k(f(x)).$$

Over usual schemes quasi-coverings and pseudo quasi-finite morphisms are equivalent notions. More generally we have the following.

Proposition 2.12. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS. If f is a quasi-covering, then is pseudo quasi-finite. Furthermore, if f is adic the converse holds.*

Proof. Suppose that f is a quasi-covering and let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. For $x \in \mathfrak{X}$ and $y = f(x)$, $\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},y}} k(y)$ is a finite $k(y)$ -module and, therefore,

$$\frac{\mathcal{O}_{X_0,x}}{\mathfrak{m}_{Y_0,y}\mathcal{O}_{X_0,x}} = \frac{\mathcal{O}_{\mathfrak{X},x}}{\mathcal{J}\mathcal{O}_{\mathfrak{X},x}} \otimes_{\mathcal{O}_{Y_0,y}} k(y)$$

is $k(y)$ -finite, so it follows that f is pseudo quasi-finite.

If f is an adic morphism, $f^{-1}(y) = f_0^{-1}(y)$ for each $y \in \mathfrak{Y}$ then,

$$\mathcal{O}_{X_0,x}/\mathfrak{m}_{Y_0,y}\mathcal{O}_{X_0,x} = \mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},y}} k(y)$$

for all $x \in \mathfrak{X}$ with $y = f(x)$. If f is moreover pseudo quasi-finite, it follows from [EGA I, Corollaire (6.11.2)] that f is a quasi-covering. □

Corollary 2.13. *Every finite morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS is a quasi-covering.*

Proof. Finite morphisms are adic and pseudo quasi-finite. Therefore the result is consequence of the last proposition. □

Nevertheless, by the next example, not every pseudo finite morphism is a quasi-covering and, therefore, *pseudo quasi-finite* does not imply *quasi-covering* for morphisms in NFS.

Example 2.14. For $r > 0$, the canonical projection $p : \mathbb{D}_{\mathfrak{X}}^r \rightarrow \mathfrak{X}$ is not a quasi-covering since

$$\dim_x p \underset{1.15.(1)}{=} r > 0 \quad \forall x \in \mathfrak{X}.$$

But considering an appropriate pair of ideals of definition, the scheme map $p_0 = 1_{X_0}$ is finite.

2.15. In short, we have the following diagram of strict implications (with the conditions that imply adic morphism in italics):

$$\begin{array}{ccccc}
 \textit{closed immersion} & \Rightarrow & \textit{finite} & \Rightarrow & \text{quasi-covering} \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{pseudo closed immersion} & \Rightarrow & \text{pseudo finite} & \Rightarrow & \text{pseudo quasi-finite}
 \end{array}$$

3. FLAT MORPHISMS AND COMPLETION MORPHISMS

In the first part of this section we discuss flat morphisms in NFS. Whenever a morphism

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

is adic, the local criterion of flatness for formal schemes (Proposition 3.3) relates the flat character of f and that of the morphisms f_n , for all $n \in \mathbb{N}$. In absence of the adic hypothesis this relation does not hold, though (Example 3.2). In the second part, we study the morphisms of completion in NFS, a class of flat morphisms that are pseudo closed immersions (so, they are closed immersions as topological maps). Despite the construction of the completion of a formal scheme along a closed formal subscheme is clearly natural in NFS, it has not been systematically developed in the basic references about formal schemes. Morphisms of completion will be an essential ingredient in the main theorems of Section 7, namely, Theorems 7.11, 7.12 and 7.13.

3.1. A morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is *flat at* $x \in \mathfrak{X}$ if $\mathcal{O}_{\mathfrak{X},x}$ is a flat $\mathcal{O}_{\mathfrak{Y},f(x)}$ -module. We say that f is *flat* if it is flat at x , for all $x \in \mathfrak{X}$.

Given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, by [B, III, §5.4, Proposition 4] the following are equivalent:

- (1) f is flat at $x \in \mathfrak{X}$
- (2) $\mathcal{O}_{\mathfrak{X},x}$ is a flat $\mathcal{O}_{\mathfrak{Y},f(x)}$ -module
- (3) $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a flat $\widehat{\mathcal{O}_{\mathfrak{Y},f(x)}}$ -module
- (4) $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a flat $\widehat{\mathcal{O}_{\mathfrak{Y},f(x)}}$ -module

Example 3.2. If K is a field and $\mathbb{A}_K^1 = \text{Spec}(K[T])$ consider the closed subset $X' = V(\langle T \rangle) \subset \mathbb{A}_K^1$. The canonical morphism of completion of \mathbb{A}_K^1 along X'

$$\mathbb{D}_K^1 \xrightarrow{\kappa} \mathbb{A}_K^1$$

is flat but, the morphisms

$$\text{Spec}(K[T]/\langle T \rangle^{n+1}) \xrightarrow{\kappa_n} \mathbb{A}_K^1$$

are not flat, for every $n \in \mathbb{N}$.

Proposition 3.3. (*Local flatness criterion for formal schemes*) Given an adic morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS, $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$, let $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$ be Ideals of definition and let $\{f_n : X_n \rightarrow Y_n\}_{n \in \mathbb{N}}$ be the morphisms induced by f and \mathcal{K} . The following assertions are equivalent:

- (1) The morphism f is flat.

- (2) The morphism f_n is flat, for all $n \in \mathbb{N}$.
- (3) The morphism f_0 is flat.

Proof. We may suppose that $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is in NFS. Then if $\mathcal{K} = K^\Delta$ for an ideal of definition $K \subset B$, we have that $\mathcal{J} = (KA)^\Delta$ and the proposition is a consequence of [AJL1, Lemma 7.1.1] and of the local flatness criterion for rings (*cf.* [M2, Theorem 22.3]). \square

Associated to a (usual) locally noetherian scheme X and a closed subscheme of $X' \subset X$ there is a locally noetherian formal scheme $X_{/X'}$, called completion of X along X' and, a canonical morphism $\kappa : X_{/X'} \rightarrow X$ ([EGA I, (10.8.3) and (10.8.5)]). Next, we define the completion of a formal scheme \mathfrak{X} along a closed formal subscheme $\mathfrak{X}' \subset \mathfrak{X}$.

Definition 3.4. Let \mathfrak{X} be in NFS and $\mathfrak{X}' \subset \mathfrak{X}$ a closed formal subscheme defined by a coherent Ideal \mathcal{I} of $\mathcal{O}_{\mathfrak{X}}$. Given an Ideal of definition \mathcal{J} of \mathfrak{X} We define the completion of a sheaf \mathcal{F} on \mathfrak{X} over \mathfrak{X}' , and it will be denoted $\mathcal{F}_{/\mathfrak{X}'}$, as the restriction to \mathfrak{X}' of the sheaf

$$\varprojlim_{n \in \mathbb{N}} \frac{\mathcal{F}}{(\mathcal{J} + \mathcal{I})^{n+1} \mathcal{F}}.$$

The definition does not depend neither on the chosen Ideal of definition \mathcal{J} of \mathfrak{X} nor on the coherent Ideal \mathcal{I} that defines \mathfrak{X}' .

We define the *completion of \mathfrak{X} along \mathfrak{X}'* , and it will be denoted $\mathfrak{X}_{/\mathfrak{X}'}$, as the topological ringed space whose underlying topological space is \mathfrak{X}' and whose sheaf of topological rings is $\mathcal{O}_{\mathfrak{X}_{/\mathfrak{X}'}}$.

It is easy to check that $\mathfrak{X}_{/\mathfrak{X}'}$ satisfies the hypothesis of [EGA I, (10.6.3) and (10.6.4)], from which we deduce that:

- (1) The formal scheme $\mathfrak{X}_{/\mathfrak{X}'}$ is locally noetherian.
- (2) The Ideal $(\mathcal{I} + \mathcal{J})_{/\mathfrak{X}'} \subset \mathcal{O}_{\mathfrak{X}_{/\mathfrak{X}'}}$ defined by the restriction to \mathfrak{X}' of the sheaf

$$\varprojlim_{n \in \mathbb{N}} \frac{(\mathcal{J} + \mathcal{I})}{(\mathcal{J} + \mathcal{I})^{n+1}}$$

is an Ideal of definition of $\mathfrak{X}_{/\mathfrak{X}'}$.

- (3) It holds that $\mathcal{O}_{\mathfrak{X}_{/\mathfrak{X}'}} / ((\mathcal{I} + \mathcal{J})_{/\mathfrak{X}'})^{n+1}$ agrees with the restriction to \mathfrak{X}' of the sheaf $\mathcal{O}_{\mathfrak{X}} / (\mathcal{J} + \mathcal{I})^{n+1}$.

3.5. With the above notations, if $Z_n = (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}} / (\mathcal{J} + \mathcal{I})^{n+1})$ for all $n \in \mathbb{N}$, by 1.1 we have that

$$\mathfrak{X}_{/\mathfrak{X}'} = \varprojlim_{n \in \mathbb{N}} Z_n$$

For each $n \in \mathbb{N}$, let $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{J}^{n+1})$ and $X'_n = (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}} / (\mathcal{J}^{n+1} + \mathcal{I}))$. The canonical morphisms

$$\frac{\mathcal{O}_{\mathfrak{X}}}{\mathcal{J}^{n+1}} \twoheadrightarrow \frac{\mathcal{O}_{\mathfrak{X}}}{(\mathcal{J} + \mathcal{I})^{n+1}} \twoheadrightarrow \frac{\mathcal{O}_{\mathfrak{X}}}{\mathcal{J}^{n+1} + \mathcal{I}}$$

provide the closed immersions of schemes $X'_n \xrightarrow{j_n} Z_n \xrightarrow{\kappa_n} X_n$, for all $n \in \mathbb{N}$ such that the following diagrams are commutative:

$$\begin{array}{ccccc} X'_m & \xrightarrow{j_m} & Z_m & \xrightarrow{\kappa_m} & X_m \\ \uparrow & & \uparrow & & \uparrow \\ X'_n & \xrightarrow{j_n} & Z_n & \xrightarrow{\kappa_n} & X_n \end{array}$$

for all $m \geq n \geq 0$. Then by 1.1 we have the canonical morphisms in NFS

$$\mathfrak{X}' \xrightarrow{j} \mathfrak{X}_{/\mathfrak{X}'} \xrightarrow{\kappa} \mathfrak{X}$$

where j is a closed immersion (see 1.5). The morphism κ as topological map is the inclusion and it is called *morphism of completion of \mathfrak{X} along \mathfrak{X}'* .

Remark. Observe that κ is adic only if \mathcal{I} is contained in a Ideal of definition of \mathfrak{X} , in which case $\mathfrak{X} = \mathfrak{X}_{/\mathfrak{X}'}$ and $\kappa = 1_{\mathfrak{X}}$.

3.6. If $\mathfrak{X} = \mathrm{Spf}(A)$ is in $\mathrm{NFS}_{\mathrm{af}}$ with A a J -adic noetherian ring, and $\mathfrak{X}' = \mathrm{Spf}(A/I)$ is a closed formal scheme of \mathfrak{X} , then

$$\Gamma(\mathfrak{X}_{/\mathfrak{X}'}, \mathcal{O}_{\mathfrak{X}_{/\mathfrak{X}'}}) = \varprojlim_{n \in \mathbb{N}} \frac{A}{(J+I)^{n+1}} =: \widehat{A}$$

and from [EGA I, (10.2.2) and (10.4.6)] we have that $\mathfrak{X}_{/\mathfrak{X}'} = \mathrm{Spf}(\widehat{A})$ and the morphisms $\mathfrak{X}' \xrightarrow{j} \mathfrak{X}_{/\mathfrak{X}'} \xrightarrow{\kappa} \mathfrak{X}$ correspond to the natural continuous morphisms $A \rightarrow \widehat{A} \rightarrow A/I$.

Proposition 3.7. *Given \mathfrak{X} in NFS and \mathfrak{X}' a closed formal subscheme of \mathfrak{X} , the morphism of completion $\kappa : \mathfrak{X}_{/\mathfrak{X}'} \rightarrow \mathfrak{X}$ is a pseudo closed immersion and étale (and therefore, from [AJP, Proposition 4.8], it is flat).*

Proof. With the notations of 3.5 we have that

$$\kappa = \varprojlim_{n \in \mathbb{N}} \kappa_n.$$

Since κ_n is a closed immersion for all $n \in \mathbb{N}$, it follows that κ is a pseudo closed immersion. In order to prove that κ is an étale morphism we may suppose that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{X}' = \mathrm{Spf}(A/I)$, where A is a J -adic noetherian ring. Note that $\mathfrak{X}_{/\mathfrak{X}'} = \mathrm{Spf}(\widehat{A})$ where \widehat{A} is the completion of A for the $(J+I)$ -adic topology and, therefore, is étale over A . By [AJP, 2.2], κ is an étale morphism. \square

Remark. In Theorem 7.5 we will see that the converse holds: every flat pseudo closed immersion is a morphism of completion.

3.8. Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS, let $\mathfrak{X}' \subset \mathfrak{X}$ and $\mathfrak{Y}' \subset \mathfrak{Y}$ be closed formal subschemes given by Ideals $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{L} \subset \mathcal{O}_{\mathfrak{Y}}$ such that $f^*(\mathcal{L})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$,

that is, $f(\mathfrak{X}') \subset \mathfrak{Y}'$. If $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ are Ideals of definition of \mathfrak{X} and \mathfrak{Y} , respectively, such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. Let us denote for all $n \in \mathbb{N}$

$$\begin{aligned} X_n &= (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1}), & Y_n &= (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^{n+1}), \\ Z_n &= (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}}/(\mathcal{J} + \mathcal{I})^{n+1}), & W_n &= (\mathfrak{Y}', \mathcal{O}_{\mathfrak{Y}}/(\mathcal{K} + \mathcal{L})^{n+1}) \\ X'_n &= (\mathfrak{X}', \mathcal{O}_{\mathfrak{X}}/(\mathcal{J}^{n+1} + \mathcal{I})) & Y'_n &= (\mathfrak{Y}', \mathcal{O}_{\mathfrak{X}}/(\mathcal{K}^{n+1} + \mathcal{L})). \end{aligned}$$

Then the morphism f induces the following commutative diagram of locally noetherian schemes:

$$\begin{array}{ccccc} X_n & \xrightarrow{f_n} & Y_n & & \\ \uparrow \kappa_n & \searrow & \uparrow & \searrow & \\ & X_m & \xrightarrow{f_m} & Y_m & \\ & \uparrow \kappa_m & \uparrow \kappa'_m & \uparrow \kappa'_m & \\ Z_n & \xrightarrow{\widehat{f}_n} & W_n & & \\ \uparrow j_n & \searrow & \uparrow \widehat{f}_m & \searrow & \\ & Z_m & \xrightarrow{\widehat{f}_m} & W_m & \\ & \uparrow j_m & \uparrow j'_m & \uparrow j'_m & \\ X'_n & \xrightarrow{f'_n} & Y'_n & & \\ & \searrow & \uparrow & \searrow & \\ & X'_m & \xrightarrow{f'_m} & Y'_m & \end{array}$$

for all $m \geq n \geq 0$. Note that

$$f' = \varinjlim_{n \in \mathbb{N}} f'_n$$

is the restriction $f|_{\mathfrak{X}'}: \mathfrak{X}' \rightarrow \mathfrak{Y}'$. Applying the direct limit over $n \in \mathbb{N}$ we obtain a morphism

$$\widehat{f}: \mathfrak{X}/_{\mathfrak{X}'} \rightarrow \mathfrak{Y}/_{\mathfrak{Y}'}$$

in NFS, such that the following diagram is commutative:

$$(3.8.1) \quad \begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \uparrow \kappa & & \uparrow \kappa' \\ \mathfrak{X}/_{\mathfrak{X}'} & \xrightarrow{\widehat{f}} & \mathfrak{Y}/_{\mathfrak{Y}'} \\ \uparrow & & \uparrow \\ \mathfrak{X}' & \xrightarrow{f|_{\mathfrak{X}'}} & \mathfrak{Y}' \end{array}$$

We will call \widehat{f} the *completion of f along \mathfrak{X}' and \mathfrak{Y}'* .

3.9. Suppose that $f: \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathrm{NFS}_{\mathrm{af}}$ and that $\mathfrak{X}' = \mathrm{Spf}(A/I)$ and $\mathfrak{Y}' = \mathrm{Spf}(B/L)$ with $LA \subset I$. If $J \subset A$ and $K \subset B$

are Ideals of definition such that $KA \subset J$, the morphism $\widehat{f} : \mathfrak{X}_{/\mathfrak{X}'} \rightarrow \mathfrak{Y}_{/\mathfrak{Y}'}$ corresponds to the morphism induced by $B \rightarrow A$

$$\widehat{B} \rightarrow \widehat{A}$$

(cf. [EGA I, (10.4.6)]) where \widehat{A} is the completion of A for the $(I+J)$ -adic topology and \widehat{B} denotes the completion of B for the $(K+L)$ -adic topology.

Proposition 3.10. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS, let $\mathfrak{Y}' \subset \mathfrak{Y}$ be a closed formal subscheme and $\mathfrak{X}' = f^{-1}(\mathfrak{Y}')$. Then,*

$$\mathfrak{X}_{/\mathfrak{X}'} = \mathfrak{Y}_{/\mathfrak{Y}'} \times_{\mathfrak{Y}} \mathfrak{X}.$$

Proof. We may restrict to the case in which $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ and $\mathfrak{Y}' = \mathrm{Spf}(B/L)$ are affine formal schemes and $J \subset A$ and $K \subset B$ are ideals of definition such that $KA \subset J$. By hypothesis, $\mathfrak{X}' = \mathrm{Spf}(A/LA)$, so $\mathfrak{X}_{/\mathfrak{X}'} = \mathrm{Spf}(\widehat{A})$ where \widehat{A} is the completion of A for the $(J+LA)$ -adic topology. On the other hand, $\mathfrak{Y}_{/\mathfrak{Y}'} = \mathrm{Spf}(\widehat{B})$ where \widehat{B} denotes the completion of B for the $(K+L)$ -adic topology and it holds that

$$\widehat{B} \widehat{\otimes}_B A = B \widehat{\otimes}_B A = \widehat{A},$$

since $J + (K+L)A = J + KA + LA = J + LA$ so we get the result. \square

Proposition 3.11. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS, consider $\mathfrak{X}' \subset \mathfrak{X}$ and $\mathfrak{Y}' \subset \mathfrak{Y}$ closed formal subschemes such that $f(\mathfrak{X}') \subset \mathfrak{Y}'$.*

(1) *Let \mathcal{P} be one of the following properties of morphisms in NFS:*

pseudo finite type, pseudo finite, pseudo closed immersion, pseudo quasi-finite, quasi-covering, flat, separated, radical, smooth, unramified, étale.

If f satisfies \mathcal{P} , then so does \widehat{f} .

(2) *Moreover, if $\mathfrak{X}' = f^{-1}(\mathfrak{Y}')$, let \mathcal{Q} be one of the following properties of morphisms in NFS:*

adic, finite type, finite, closed immersion, smooth adic, unramified adic, étale adic.

Then, if f satisfies \mathcal{Q} , then so does \widehat{f} .

Proof. Suppose that f is flat and let us prove that \widehat{f} is flat. The question is local so we may assume $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ in $\mathrm{NFS}_{\mathrm{af}}$, $\mathfrak{X}' = \mathrm{Spf}(A/I)$ and $\mathfrak{Y}' = \mathrm{Spf}(B/L)$ with $LA \subset I$. Let $J \subset A$ and $K \subset B$ be ideals of definition such that $KA \subset J$ and, \widehat{A} and \widehat{B} the completions of A and B for the topologies given by $(I+J) \subset A$ and $(K+L) \subset B$, respectively. By [B, III, §5.4, Proposition 4] we have that the morphism $\widehat{B} \rightarrow \widehat{A}$ is flat and, from 3.9 and [AJL1, Lemma 7.1.1] there results that \widehat{f} is flat.

Suppose that f satisfies any of the other properties \mathcal{P} and let us prove that \widehat{f} inherits it, using the commutativity of the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \kappa \uparrow & & \uparrow \kappa' \\ \mathfrak{X}/\mathfrak{X}' & \xrightarrow{\widehat{f}} & \mathfrak{Y}/\mathfrak{Y}' \end{array}$$

where the vertical arrows are morphisms of completion. Since all of this properties \mathcal{P} are stable under composition and a morphism of completion satisfies \mathcal{P} (Proposition 3.7) we have that \mathcal{P} holds for $f \circ \kappa = \kappa' \circ \widehat{f}$. If \mathcal{P} is smooth, unramified or étale the result is immediate from [AJP, Proposition 2.13].

If \mathcal{P} is any of the other properties, then closed immersions verify \mathcal{P} and \mathcal{P} is stable under composition and under base-change in NFS. Therefore, since $\kappa' \circ \widehat{f}$ has \mathcal{P} and κ' is separated (Proposition 3.7), by the analogous argument in NFS to the one in Sch [EGA I, (5.2.7), *i*), *ii*) \Rightarrow *iii*)] we get that \widehat{f} also satisfies \mathcal{P} .

Finally, if f is adic, from Proposition 3.10 and from [AJP, 1.3], we deduce that \widehat{f} is adic. Then, if \mathcal{Q} is any of the properties in statement (2) and f satisfies \mathcal{Q} , by (1) so does \widehat{f} . \square

4. UNRAMIFIED MORPHISMS

Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of locally noetherian formal schemes. Given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, let us express f as a limit

$$f : \mathfrak{X} \rightarrow \mathfrak{Y} = \varinjlim_{n \in \mathbb{N}} (f_n : X_n \rightarrow Y_n).$$

We begin relating the unramified character of $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ and that of the underlying ordinary scheme morphisms $\{f_n\}_{n \in \mathbb{N}}$.

Proposition 4.1. *With the previous notations, the morphism f is unramified if, and only if, $f_n : X_n \rightarrow Y_n$ is unramified, for all $n \in \mathbb{N}$.*

Proof. Applying [AJP, Proposition 4.6] we have to show that $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = 0$ is equivalent to $\Omega_{X_n/Y_n}^1 = 0$, for all $n \in \mathbb{N}$. If $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = 0$, by the Second Fundamental Exact Sequence ([AJP, Proposition 3.13]) for the morphisms

$$X_n \hookrightarrow \mathfrak{X} \xrightarrow{f} \mathfrak{Y},$$

we have that $\Omega_{X_n/\mathfrak{Y}}^1 = 0$, for all $n \in \mathbb{N}$. From the First Fundamental Exact Sequence ([AJP, Proposition 3.10]) associated to the morphisms

$$X_n \xrightarrow{f_n} Y_n \hookrightarrow \mathfrak{Y},$$

there results that $\Omega_{X_n/Y_n}^1 = 0$. The converse follows from the identification

$$\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = \varprojlim_{n \in \mathbb{N}} \Omega_{X_n/Y_n}^1$$

(cfr. [AJP, §1.9]). □

Corollary 4.2. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS and let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. If the induced morphisms $f_n : X_n \rightarrow Y_n$ are immersions for all $n \in \mathbb{N}$, then f is unramified.*

In the class of adic morphisms in NFS the following proposition provides a criterion, stronger than the last result, to determine when a morphism f is unramified.

Proposition 4.3. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an adic morphism in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition. Write*

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

by taking Ideals of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$. The morphism f is unramified if, and only if, the induced morphism $f_0 : X_0 \rightarrow Y_0$ is unramified.

Proof. If f is unramified by Proposition 4.1 we have that f_0 is unramified. Conversely, suppose that f_0 is unramified and let us prove that $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = 0$. The question is local so we may assume that $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathrm{NFS}_{\mathrm{af}}$ and that $\mathcal{J} = J^\Delta$, with $J \subset A$ an ideal of definition. By hypothesis $\Omega_{X_0/Y_0}^1 = 0$ and thus, since f is adic there results that

$$(4.3.1) \quad \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_0} \underset{[\mathrm{AJP}, 3.8]}{=} \Omega_{X_0/Y_0}^1 = 0.$$

Then by the equivalence of categories [EGA I, (10.10.2)], the last equality says that $\widehat{\Omega}_{A/B}^1/J\widehat{\Omega}_{A/B}^1 = 0$. Since A is a J -adic ring it holds that J is contained in the Jacobson radical of A . Moreover, [AJP, Proposition 3.3] implies that $\widehat{\Omega}_{A/B}^1$ is a finite type A -module. From Nakayama's lemma we deduce that $\widehat{\Omega}_{A/B}^1 = 0$ and therefore, $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = (\widehat{\Omega}_{A/B}^1)^\Delta = 0$. Applying [AJP, Proposition 4.6] it follows that f is unramified. □

The following example illustrates that in the non adic case the analogous of the last proposition does not hold.

Example 4.4. Let K be a field and $p : \mathbb{D}_K^1 \rightarrow \mathrm{Spec}(K)$ be the projection morphism of the formal disc of dimension 1 over $\mathrm{Spec}(K)$. By [AJP, Example 3.14] we have that $\widehat{\Omega}_p^1 = (K[[T]]\widehat{dT})^\Delta$ and therefore, \mathbb{D}_K^1 is ramified over K ([AJP, Proposition 4.6]). However, given the ideal of definition $\langle T \rangle \subset K[[T]]$ the induced morphism $p_0 = 1_{\mathrm{Spec}(K)}$ is unramified.

Let us consider for a morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS the notation established at the beginning of the section. In view of the example, our next goal will be to determine when the morphism f such that f_0 is unramified but f itself is not necessarily adic, is unramified (Corollary 4.10). In order to do that, we will need some results that describe the local behavior of unramified morphisms. Next, we provide local characterizations of unramified morphisms in NFS, generalizing the analogous properties in the category of schemes (cf. [EGA IV₄, (17.4.1)]).

Proposition 4.5. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS of pseudo finite type. Given $x \in \mathfrak{X}$ and $y = f(x)$ the following conditions are equivalent:*

- (1) f is unramified at x .
- (2) $f^{-1}(y)$ is an unramified $k(y)$ -formal scheme at x .
- (3) $\mathfrak{m}_{\mathfrak{X},x}\widehat{\mathcal{O}_{\mathfrak{X},x}} = \mathfrak{m}_{\mathfrak{Y},y}\widehat{\mathcal{O}_{\mathfrak{X},x}}$ and $k(x)|k(y)$ is a finite separable extension.
- (4) $\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 = 0$
- (4') $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x = 0$
- (5) $\mathcal{O}_{\mathfrak{X},x}$ is a formally unramified $\mathcal{O}_{\mathfrak{Y},y}$ -algebra for the adic topologies.
- (5') $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally unramified $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ -algebra for the adic topologies.

Proof. Let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ which allows us to write

$$f : \mathfrak{X} \rightarrow \mathfrak{Y} = \varprojlim_{n \in \mathbb{N}} (f_n : X_n \rightarrow Y_n)$$

(1) \Leftrightarrow (2) By Proposition 4.1, f is unramified at x if, and only if, all the morphisms $f_n : X_n \rightarrow Y_n$ are unramified at x . Applying [EGA IV₄, (17.4.1)], this is equivalent to $f_n^{-1}(y)$ being an unramified $k(y)$ -scheme at x , for all $n \in \mathbb{N}$, which is also equivalent to

$$f^{-1}(y) \underset{1.12}{=} \varprojlim_{n \in \mathbb{N}} f_n^{-1}(y)$$

being an unramified $k(y)$ -formal scheme at x .

(1) \Leftrightarrow (3) The assertion (1) is equivalent to $f_n : X_n \rightarrow Y_n$ being unramified at x , for all $n \in \mathbb{N}$, and from [EGA IV₄, *loc. cit.*] there results that $k(x)|k(y)$ is a finite separable extension, and that $\mathfrak{m}_{X_n,x} = \mathfrak{m}_{Y_n,y}\mathcal{O}_{X_n,x}$, for all $n \in \mathbb{N}$. Hence,

$$\mathfrak{m}_{\mathfrak{X},x}\widehat{\mathcal{O}_{\mathfrak{X},x}} = \varprojlim_{n \in \mathbb{N}} \mathfrak{m}_{X_n,x} = \varprojlim_{n \in \mathbb{N}} \mathfrak{m}_{Y_n,y}\mathcal{O}_{X_n,x} = \mathfrak{m}_{\mathfrak{Y},y}\widehat{\mathcal{O}_{\mathfrak{X},x}}.$$

(4) \Leftrightarrow (4') By [AJP, Proposition 3.3] it holds that $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x$ is a finite type $\mathcal{O}_{\mathfrak{X},x}$ -module and therefore,

$$\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 = (\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x = (\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x \otimes_{\mathcal{O}_{\mathfrak{X},x}} \widehat{\mathcal{O}_{\mathfrak{X},x}}.$$

Then, since $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a faithfully flat $\mathcal{O}_{\mathfrak{X},x}$ -algebra, $\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 = 0$ if, and only if, $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x = 0$.

(3) \Rightarrow (4) Since $k(x)|k(y)$ is a finite separable extension we have that $\Omega_{k(x)/k(y)}^1 = 0$ and from [AJP, Proposition 3.3] $\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 = (\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x$ is a finite type $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ -module. Therefore, it holds that

$$\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 \otimes_{\widehat{\mathcal{O}_{\mathfrak{X},x}}} k(x) = \widehat{\Omega}_{(\mathcal{O}_{\mathfrak{X},x} \otimes_{\mathcal{O}_{\mathfrak{Y},y}} k(y))/k(y)}^1 = \Omega_{k(x)/k(y)}^1 = 0.$$

By Nakayama's lemma, $\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1 = 0$.

(4) \Leftrightarrow (5) It is straightforward from [EGA IV₁, (0, 20.7.4)].

(5) \Leftrightarrow (5') Immediate.

(4') \Rightarrow (1) Since $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \in \text{Coh}(\mathfrak{X})$ ([AJP, Proposition 3.3]), assertion (4') implies that there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)|_{\mathfrak{U}} = 0$ and therefore, by [AJP, Proposition 4.6] we have that f is unramified at x . \square

Corollary 4.6. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. The following conditions are equivalent:*

- (1) f is unramified.
- (2) For all $x \in \mathfrak{X}$, $f^{-1}(f(x))$ is an unramified $k(f(x))$ -formal scheme at x .
- (3) For all $x \in \mathfrak{X}$, $\mathfrak{m}_{\mathfrak{X},x} \widehat{\mathcal{O}_{\mathfrak{X},x}} = \mathfrak{m}_{\mathfrak{Y},f(x)} \widehat{\mathcal{O}_{\mathfrak{X},x}}$ and $k(x)|k(f(x))$ is a finite separable extension.
- (4) $\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},f(x)}}^1 = 0$, for all $x \in \mathfrak{X}$.
- (4') For all $x \in \mathfrak{X}$, $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x = 0$.
- (5) For all $x \in \mathfrak{X}$, $\mathcal{O}_{\mathfrak{X},x}$ is a formally unramified $\mathcal{O}_{\mathfrak{Y},f(x)}$ -algebra for the adic topologies.
- (5') For all $x \in \mathfrak{X}$, $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally unramified $\widehat{\mathcal{O}_{\mathfrak{Y},f(x)}}$ -algebra for the adic topologies.

Corollary 4.7. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. If f is unramified at $x \in \mathfrak{X}$, then f is a quasi-covering at x .*

Proof. By assertion (3) of Proposition 4.5 we have that

$$\mathcal{O}_{\mathfrak{X},x} \widehat{\otimes}_{\mathcal{O}_{\mathfrak{Y},f(x)}} k(f(x)) = k(x)$$

with $k(x)|k(f(x))$ a finite extension and therefore, f is a quasi-covering at x (see Definition 2.8). \square

Corollary 4.8. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. If f is unramified at $x \in \mathfrak{X}$, then $\dim_x f = 0$.*

Proof. It is straightforward from the previous Corollary and Proposition 2.11. \square

Proposition 4.9. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. Given $x \in \mathfrak{X}$ and $y = f(x)$ the following conditions are equivalent:*

- (1) f is unramified at x

(2) $f_0 : X_0 \rightarrow Y_0$ is unramified at x and $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) = k(x)$

Proof. If f is unramified at x , then f_0 is unramified at x (Proposition 4.1). Moreover, assertion (3) of Proposition 4.5 implies that $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) = k(x)$ so (1) \Rightarrow (2) holds. Let us prove that (2) \Rightarrow (1). Since f_0 is unramified at x we have that $k(x)|k(y)$ is a finite separable extension (cf. [EGA IV₄, (17.4.1)]). From the equality $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) = k(x)$ we deduce that $\mathfrak{m}_{\mathfrak{X},x} \widehat{\mathcal{O}_{\mathfrak{X},x}} = \mathfrak{m}_{\mathfrak{Y},y} \widehat{\mathcal{O}_{\mathfrak{X},x}}$. Thus, the morphism f and the point x satisfy assertion (3) of Proposition 4.5 and there results that f is unramified at x . \square

Now we are ready to state the non adic version of Proposition 4.3:

Corollary 4.10. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism in NFS of pseudo finite type let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and let $f_0 : X \rightarrow Y$ be the induced morphism. The following conditions are equivalent:*

- (1) *The morphism f is unramified.*
- (2) *The morphism f_0 is unramified and, for all $x \in \mathfrak{X}$, $f^{-1}(y) = f_0^{-1}(y)$ with $y = f(x)$.*

Proof. Suppose that f is unramified and fix $x \in \mathfrak{X}$ and $y = f(x)$. By Proposition 4.9 we have that f_0 is unramified and that $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) = k(x)$. Therefore, $\mathcal{J} \cdot (\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)) = 0$ and applying Lemma 4.11 we deduce that $f^{-1}(y) = f_0^{-1}(y)$. Reciprocally, suppose that (2) holds and let us show that given $x \in \mathfrak{X}$, the morphism f is unramified at x . If $y = f(x)$, we have that $f_0^{-1}(y)$ is an unramified $k(y)$ -scheme at x (cf. [EGA IV₄, (17.4.1)]) and since $f^{-1}(y) = f_0^{-1}(y)$, from Proposition 4.5 there results that f is unramified at x . \square

Lemma 4.11. *Let A be a J -adic noetherian ring such that for all open prime ideals $\mathfrak{p} \subset A$, $J_{\mathfrak{p}} = 0$. Then $J = 0$ and therefore, the J -adic topology in A is the discrete topology.*

Proof. Since every maximal ideal $\mathfrak{m} \subset A$ is open for the J -adic topology, we have that $J_{\mathfrak{m}} = 0$, for all maximal ideal $\mathfrak{m} \subset A$, so $J = 0$. \square

4.12. As a consequence of Corollary 4.10 there results that:

- If $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is an unramified morphism in NFS then $f^{-1}(y)$ is a usual scheme for all $x \in \mathfrak{X}$ being $y = f(x)$.
- In Corollary 4.6 assertion (2) may be written:
 - (2') *For all $x \in \mathfrak{X}$, $y = f(x)$, $f^{-1}(y)$ is a unramified $k(y)$ -scheme at x .*

From Proposition 4.5 we obtain the following result, in which we provide a description of pseudo closed immersions that will be used in the characterization of completion morphisms (Theorem 7.5).

Corollary 4.13. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS, let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and that let us express*

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

The morphism f is a pseudo closed immersion if, and only if, f is unramified and $f_0 : X_0 \rightarrow Y_0$ is a closed immersion.

Proof. If f is a pseudo closed immersion, by Corollary 4.2 there results that f is unramified. Reciprocally, suppose that f is unramified and that f_0 is a closed immersion and let us show that $f_n : X_n \rightarrow Y_n$ is a closed immersion, for each $n \in \mathbb{N}$. By [EGA I, (4.2.2.(ii))] it suffices to prove that, for all $x \in \mathfrak{X}$ with $y = f(x)$, the morphism $\mathcal{O}_{Y_n, y} \rightarrow \mathcal{O}_{X_n, x}$ is surjective, for all $n \in \mathbb{N}$. Fix $x \in \mathfrak{X}$, $y = f(x) \in \mathfrak{Y}$ and $n \in \mathbb{N}$. Since f_0 is a closed immersion, by [EGA I, *loc. cit.*], we have that $\mathcal{O}_{Y_0, y} \rightarrow \mathcal{O}_{X_0, x}$ is surjective and therefore, $\mathrm{Spf}(\widehat{\mathcal{O}_{\mathfrak{X}, x}}) \rightarrow \mathrm{Spf}(\widehat{\mathcal{O}_{\mathfrak{Y}, y}})$ is a pseudo finite morphism, so, the morphism $\mathcal{O}_{Y_n, y} \rightarrow \mathcal{O}_{X_n, x}$ is finite. On the other hand, the morphism f is unramified therefore by Proposition 4.1 we get that f_n is unramified and applying Proposition 4.5 we obtain that $\mathfrak{m}_{Y_n, y}\mathcal{O}_{X_n, x} = \mathfrak{m}_{X_n, x}$. Then by Nakayama's lemma we conclude that $\mathcal{O}_{Y_n, y} \rightarrow \mathcal{O}_{X_n, x}$ is a surjective morphism. \square

5. SMOOTH MORPHISMS

The contents of this section can be structured in two parts. In the first part we study the relationship between the smoothness of a morphism

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

in NFS and the smoothness of the ordinary scheme morphisms $\{f_n\}_{n \in \mathbb{N}}$. In the second part, we provide a local factorization for smooth morphisms (Proposition 5.9). In this section we also prove in Corollary 5.13 the matrix Jacobian criterion, that is a useful explicit condition in terms of a matrix rank for determining whether a closed subscheme of the affine formal space or of the affine formal disc is smooth or not.

Proposition 5.1. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and let us write*

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

If $f_n : X_n \rightarrow Y_n$ is smooth, for all $n \in \mathbb{N}$, then f is smooth.

Proof. By [AJP, Proposition 4.1] we may assume that f is in $\mathrm{NFS}_{\mathrm{af}}$. Let Z be an affine scheme, $w : Z \rightarrow \mathfrak{Y}$ a morphism, $T \hookrightarrow Z$ a closed \mathfrak{Y} -subscheme given by a square zero Ideal and consider $u : T \rightarrow \mathfrak{X}$ a \mathfrak{Y} -morphism. Since f and w are morphisms of affine formal schemes we find an integer $m \geq 0$ such that $w^*(\mathcal{K}^{m+1})\mathcal{O}_Z = 0$ and $u^*(\mathcal{J}^{m+1})\mathcal{O}_T = 0$ and therefore u and w factors as $T \xrightarrow{u_m} X_m \xrightarrow{i_m} \mathfrak{X}$ and $Z \xrightarrow{w_m} Y_m \xrightarrow{i_m} \mathfrak{Y}$, respectively. Since f_m is

formally smooth, there exists a Y_m -morphism $v_m : Z \rightarrow X_m$ such that the following diagram is commutative

$$\begin{array}{ccc}
 T & \hookrightarrow & Z \\
 \downarrow u_m & \nearrow v_m & \downarrow w_m \\
 X_m & \xrightarrow{f_m} & Y_m \\
 \downarrow i_m & & \downarrow \\
 \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}.
 \end{array}$$

Thus the \mathfrak{Y} -morphism $v := i_m \circ v_m$ satisfies that $v|_T = u$ and then, f is formally smooth. Moreover, since f_0 is a finite type morphism, it holds that f is of pseudo finite type and therefore, f is smooth. \square

Corollary 5.2. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an adic morphism in NFS and consider $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition. The morphism f is smooth if, and only if, all the scheme morphisms $\{f_n : X_n \rightarrow Y_n\}_{n \in \mathbb{N}}$, determined by the Ideals of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$, are smooth.*

Proof. If f is adic, by [EGA I, (10.12.2)], we have that for each $n \in \mathbb{N}$, the diagram

$$\begin{array}{ccc}
 \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\
 \uparrow & & \uparrow \\
 X_n & \xrightarrow{f_n} & Y_n
 \end{array}$$

is a cartesian square. Then by base-change ([AJP, Proposition 2.9 (2)]) we have that f_n is smooth, for all $n \in \mathbb{N}$. The reciprocal follows from the previous proposition. \square

Next example shows us that the reciprocal of Proposition 5.1 does not hold in general.

Example 5.3. Let K be a field and $\mathbb{A}_K^1 = \text{Spec}(K[T])$. Given the closed subset $X = V(\langle T \rangle) \subset \mathbb{A}_K^1$, Proposition 3.7 implies that the canonical completion morphism

$$\mathbb{D}_K^1 \xrightarrow{\kappa} \mathbb{A}_K^1$$

of \mathbb{A}_K^1 along X is étale. However, picking in \mathbb{A}_K^1 the Ideal of definition 0, the morphisms

$$\text{Spec}(K[T]/\langle T \rangle^{n+1}) \xrightarrow{\kappa_n} \mathbb{A}_K^1$$

are not flat, whence it follows that κ_n can not be smooth for all $n \in \mathbb{N}$ (see [AJP, Proposition 4.8]).

Our next goal will be to determine the relation between smoothness of a morphism

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

and that of f_0 (Corollaries 5.6 and 5.8). In order to do that, we need to characterize smoothness locally.

Proposition 5.4. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. Given $x \in \mathfrak{X}$ and $y = f(x)$ the following conditions are equivalent:*

- (1) *The morphism f is smooth at x .*
- (2) *$\widehat{\mathcal{O}}_{\mathfrak{X},x}$ is a formally smooth $\widehat{\mathcal{O}}_{\mathfrak{Y},y}$ -algebra for the adic topologies.*
- (3) *$\widehat{\mathcal{O}}_{\mathfrak{X},x}$ is a formally smooth $\widehat{\mathcal{O}}_{\mathfrak{Y},y}$ -algebra for the adic topologies.*
- (4) *The morphism f is flat at x and $f^{-1}(y)$ is a $k(y)$ -formal scheme smooth at x .*

Proof. The question is local and f is of pseudo finite type, so we may assume that $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathrm{NFS}_{\mathrm{af}}$, with $A = B\{T_1, \dots, T_r\}[[Z_1, \dots, Z_s]]/I$ and $I \subset B' := B\{T_1, \dots, T_r\}[[Z_1, \dots, Z_s]]$ an ideal ([AJP, Proposition 1.7]). Let $\mathfrak{p} \subset A$ be the open prime ideal corresponding to x , $\mathfrak{q} \subset B'$ the open prime such that $\mathfrak{p} = \mathfrak{q}/I$ and $\mathfrak{r} \subset B$ the open prime ideal corresponding to y .

(1) \Rightarrow (3) Replacing \mathfrak{X} by a sufficiently small open neighborhood of x we may suppose that A is a formally smooth B -algebra. Then, by [EGA IV₁, (0, 19.3.5)] we have that $A_{\mathfrak{p}}$ is a formally smooth $B_{\mathfrak{r}}$ -algebra and [EGA IV₁, (0, 19.3.6)] implies that $\widehat{\mathcal{O}}_{\mathfrak{X},x} = \widehat{A}_{\mathfrak{p}}$ is a formally smooth $\widehat{\mathcal{O}}_{\mathfrak{Y},y} = \widehat{B}_{\mathfrak{r}}$ -algebra.

(2) \Leftrightarrow (3) It is a consequence of [EGA IV₁, (0, 19.3.6)].

(3) \Rightarrow (1) By [EGA IV₁, (0, 19.3.6)], assertion (3) is equivalent to $A_{\mathfrak{p}}$ being a formally smooth $B_{\mathfrak{r}}$ -algebra. Then Zariski's Jacobian criterion ([AJP, Proposition 4.14] implies that the morphism of $\widehat{A}_{\mathfrak{p}}$ -modules

$$\frac{\widehat{I}_{\mathfrak{q}}}{\widehat{I}_{\mathfrak{q}}^2} \rightarrow \Omega_{B'_{\mathfrak{q}}/B_{\mathfrak{r}}}^1 \widehat{\otimes}_{B'_{\mathfrak{q}}} A_{\mathfrak{p}}$$

is right invertible. Since $\widehat{A}_{\mathfrak{p}}$ is a faithfully flat $A_{\{\mathfrak{p}\}}$ -algebra and the $A_{\{\mathfrak{p}\}}$ -module $(\widehat{\Omega}_{B'/B}^1 \otimes_{B'} A)_{\{\mathfrak{p}\}}$ is projective (see [AJP, Proposition 4.8]), there results that the morphism

$$\left(\frac{I}{I^2} \right)_{\{\mathfrak{p}\}} \rightarrow (\widehat{\Omega}_{B'/B}^1 \otimes_{B'} A)_{\{\mathfrak{p}\}}$$

is right invertible by [EGA IV₁, (0, 19.1.14.(ii))]. From the equivalence of categories [EGA I, (10.10.2)] we find an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that the morphism

$$\left(\frac{I}{I^2} \right)^{\Delta} \rightarrow \widehat{\Omega}_{\mathbb{D}_{A_{\mathfrak{r}}^s}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathbb{D}_{A_{\mathfrak{r}}^s}}} \mathcal{O}_{\mathfrak{X}}$$

is right invertible in \mathfrak{U} . Now, by Zariski's Jacobian criterion for formal schemes ([AJP, Corollary 4.15]) it follows that f is smooth in \mathfrak{U} .

(3) \Rightarrow (4) By [EGA IV₁, (0, 19.3.8)] we have that $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally smooth $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ -algebra for the topologies given by the maximal ideals. Then it follows from [EGA IV₁, (0, 19.7.1)] that $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ -flat and by 3.1, f is flat at x . Moreover from [EGA IV₁, (0, 19.3.5)] we deduce that $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)$ is a formally smooth $k(y)$ -algebra for the adic topologies or, equivalently, by (3) \Leftrightarrow (1), $f^{-1}(y)$ is a $k(y)$ -formal scheme smooth at x .

(4) \Rightarrow (3) By 3.1 we have that $A_{\mathfrak{p}}$ is a flat $B_{\mathfrak{r}}$ -module and therefore, there results that

$$(5.4.1) \quad 0 \rightarrow \frac{I_{\mathfrak{q}}}{\mathfrak{r}I_{\mathfrak{q}}} \rightarrow \frac{B'_{\mathfrak{q}}}{\mathfrak{r}B'_{\mathfrak{q}}} \rightarrow \frac{A_{\mathfrak{p}}}{\mathfrak{r}A_{\mathfrak{p}}} \rightarrow 0$$

is an exact sequence. On the other hand, since $f^{-1}(y)$ is a $k(y)$ -formal scheme smooth at x , from (1) \Rightarrow (2) we deduce that $\widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)$ is a formally smooth $k(y)$ -algebra for the adic topologies or, equivalently by [EGA IV₁, (0, 19.3.6)], $A_{\mathfrak{p}}/\mathfrak{r}A_{\mathfrak{p}}$ is a formally smooth $k(\mathfrak{r})$ -algebra for the adic topologies. Applying Zariski's Jacobian criterion ([AJP, Proposition 4.14]), we have that the morphism

$$\frac{\widehat{I}_{\mathfrak{q}}}{\widehat{I}_{\mathfrak{q}}^2} \otimes_{B_{\mathfrak{r}}} k(\mathfrak{r}) \rightarrow (\widehat{\Omega}_{B'/B}^1)_{\mathfrak{q}} \widehat{\otimes}_{B'_{\mathfrak{q}}} A_{\mathfrak{p}} \otimes_{B_{\mathfrak{r}}} k(\mathfrak{r})$$

is right invertible. Now, since $(\widehat{\Omega}_{B'/B}^1)_{\mathfrak{q}}$ is a projective $B'_{\mathfrak{q}}$ -module (see [AJP, Proposition 4.8]) by [EGA I, (0, 6.7.2)] we obtain that

$$\frac{\widehat{I}_{\mathfrak{q}}}{\widehat{I}_{\mathfrak{q}}^2} \rightarrow \widehat{\Omega}_{B'_{\mathfrak{q}}/B_{\mathfrak{r}}}^1 \widehat{\otimes}_{B'_{\mathfrak{q}}} \widehat{A}_{\mathfrak{p}}$$

is right invertible. Then, by the Zariski's Jacobian criterion, $A_{\mathfrak{p}}$ is a formally smooth $B_{\mathfrak{r}}$ -algebra for the adic topologies or, equivalently by [EGA IV₁, (0, 19.3.6)], $\widehat{A}_{\mathfrak{p}}$ is a formally smooth $\widehat{B}_{\mathfrak{r}}$ -algebra. □

Corollary 5.5. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. The following conditions are equivalent:*

- (1) *The morphism f is smooth.*
- (2) *For all $x \in \mathfrak{X}$, $\mathcal{O}_{\mathfrak{X},x}$ is a formally smooth $\mathcal{O}_{\mathfrak{Y},f(x)}$ -algebra for the adic topologies.*
- (3) *For all $x \in \mathfrak{X}$, $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally smooth $\widehat{\mathcal{O}_{\mathfrak{Y},f(x)}}$ -algebra for the adic topologies.*
- (4) *The morphism f is flat and $f^{-1}(f(x))$ is a $k(f(x))$ -formal scheme smooth at x , for all $x \in \mathfrak{X}$.*

Corollary 5.6. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an adic morphism in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition. Put*

$$f = \varinjlim_{n \in \mathbb{N}} f_n$$

using the Ideals of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$. Then, the morphism f is smooth if, and only if, it is flat and the morphism $f_0 : X_0 \rightarrow Y_0$ is smooth.*

Proof. Since f is adic, the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \uparrow & & \uparrow \\ X_0 & \xrightarrow{f_0} & Y_0 \end{array}$$

is a cartesian square ([EGA I, (10.12.2)]). If f is smooth, by base-change there results that f_0 is smooth. Moreover by [AJP, Proposition 4.8] we have that f is flat. Reciprocally, if f is adic, by 1.12, we have that $f^{-1}(f(x)) = f_0^{-1}(f(x))$, for all $x \in \mathfrak{X}$. Therefore, since f_0 is smooth, by base-change there results that $f^{-1}(f(x))$ is a $k(f(x))$ -scheme smooth at x , for all $x \in \mathfrak{X}$ and applying Corollary 5.5 we conclude that f is smooth. \square

The upcoming example shows that the last result is not true without assuming the *adic* hypothesis for the morphism f .

Example 5.7. Given K a field, let \mathbb{P}_K^n be the n -dimensional projective space and $X \subset \mathbb{P}_K^n$ a closed subscheme that is not smooth over K . If we denote by $(\mathbb{P}_K^n)_{/X}$ the completion of \mathbb{P}_K^n along X , by Proposition 3.11 we have that the morphism

$$(\mathbb{P}_K^n)_{/X} \xrightarrow{f} \mathrm{Spec}(K)$$

is smooth but $f_0 : X \rightarrow \mathrm{Spec}(K)$ is not smooth.

Corollary 5.8. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism in NFS let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and with this choice let us express*

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

If f is flat, $f_0 : X_0 \rightarrow Y_0$ is a smooth morphism and $f^{-1}(f(x)) = f_0^{-1}(f(x))$, for all $x \in \mathfrak{X}$, then f is smooth.

Proof. Since f_0 is smooth and $f^{-1}(y) = f_0^{-1}(y)$ for all $y = f(x)$ with $x \in \mathfrak{X}$, we deduce that $f^{-1}(y)$ is a smooth $k(y)$ -scheme. Besides, by hypothesis f is flat and Corollary 5.5 implies that f is smooth. \square

Example 5.7 illustrates that the reciprocal of the last corollary does not hold.

Every morphism $f : X \rightarrow Y$ smooth in Sch is locally a composition of an étale morphism $U \rightarrow \mathbb{A}_Y^r$ and a projection $\mathbb{A}_Y^r \rightarrow Y$. Proposition 5.9

generalizes this fact for smooth morphisms in NFS. The same result has already appeared stated in local form in [Y, Proposition 1.11]. We include it here for completeness.

Proposition 5.9. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. The morphism f is smooth at $x \in \mathfrak{X}$ if, and only if, there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xrightarrow{g} \mathbb{A}_{\mathfrak{Y}}^n \xrightarrow{p} \mathfrak{Y}$$

where g is étale, p is the canonical projection and $n = \text{rg}(\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},f(x)}}^1)$.

Proof. As this is a local question, we may assume that $f : \mathfrak{X} = \text{Spf}(A) \rightarrow \mathfrak{Y} = \text{Spf}(B)$ is a smooth morphism in NFS_{af} . By [AJP, Proposition 4.8] and by [EGA I, (10.10.8.6)] we have that $\widehat{\Omega}_{A/B}^1$ is a projective A -module of finite type and therefore, if $\mathfrak{p} \subset A$ is the open prime ideal corresponding to x , there exists $h \in A \setminus \mathfrak{p}$ such that $\Gamma(\mathfrak{D}(h), \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1) = \widehat{\Omega}_{A_{\{h\}}/B}^1$ is a free $A_{\{h\}}$ -module of finite type. Put $\mathfrak{U} = \text{Spf}(A_{\{h\}})$. Given $\{\widehat{da}_1, \widehat{da}_2, \dots, \widehat{da}_n\}$ a basis of $\widehat{\Omega}_{A_{\{h\}}/B}^1$ consider the morphism of \mathfrak{Y} -formal schemes

$$\mathfrak{U} \xrightarrow{g} \mathbb{A}_{\mathfrak{Y}}^n = \text{Spf}(B\{T_1, T_2, \dots, T_n\})$$

defined by the continuous morphism of topological B -algebras

$$\begin{array}{ccc} B\{T_1, T_2, \dots, T_n\} & \rightarrow & A_{\{h\}} \\ T_i & \rightsquigarrow & a_i \end{array}$$

See [EGA I, (10.2.2) and (10.4.6)]. The morphism g satisfies that $f|_{\mathfrak{U}} = p \circ g$. Moreover, we deduce that $g^* \widehat{\Omega}_{\mathbb{A}_{\mathfrak{Y}}^n/\mathfrak{Y}}^1 \cong \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1$ (see the definition of g) and by [AJP, Corollary 4.13] we have that g is étale. \square

Corollary 5.10. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a smooth morphism at $x \in \mathfrak{X}$ and $y = f(x)$. Then*

$$\dim_x f = \text{rg}(\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1).$$

Proof. Put $n = \text{rg}(\widehat{\Omega}_{\mathcal{O}_{\mathfrak{X},x}/\mathcal{O}_{\mathfrak{Y},y}}^1)$. By Proposition 5.9 there exists $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as $\mathfrak{U} \xrightarrow{g} \mathbb{A}_{\mathfrak{Y}}^n \xrightarrow{p} \mathfrak{Y}$ where g is an étale morphism and p is the canonical projection. Applying [AJP, Proposition 4.8] we have that $f|_{\mathfrak{U}}$ and g are flat morphisms and therefore,

$$\begin{aligned} \dim_x f &= \dim \widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y) &= \dim \widehat{\mathcal{O}_{\mathfrak{X},x}} - \dim \widehat{\mathcal{O}_{\mathfrak{Y},y}} \\ \dim_x g &= \dim \widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathbb{A}_{\mathfrak{Y}}^n, g(x)}}} k(g(x)) &= \dim \widehat{\mathcal{O}_{\mathfrak{X},x}} - \dim \widehat{\mathcal{O}_{\mathbb{A}_{\mathfrak{Y}}^n, g(x)}}. \end{aligned}$$

Now, since g is unramified by Corollary 4.8 we have that $\dim_x g = 0$ and therefore $\dim_x f = \dim \widehat{\mathcal{O}_{\mathbb{A}_{\mathfrak{Y}}^n, g(x)}} - \dim \widehat{\mathcal{O}_{\mathfrak{Y},y}} = n$. \square

Proposition 5.11. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of pseudo finite type and $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ a closed immersion given by the Ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ and put $f' = f|_{\mathfrak{X}'}$. If f is smooth at $x \in \mathfrak{X}'$, $n = \dim_x f$ and $y = f(x)$ the following conditions are equivalent:*

- (1) *The morphism f' is smooth at x and $\dim_x f'^{-1}(y) = n - m$.*
- (2) *The sequence of $\mathcal{O}_{\mathfrak{X}}$ -modules*

$$0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}'} \rightarrow \widehat{\Omega}_{\mathfrak{X}'/\mathfrak{Y}}^1 \rightarrow 0$$

is exact² at x and, on a neighborhood of x , the displayed $\mathcal{O}_{\mathfrak{X}'}$ -Modules are locally free of ranks m , n and $n - m$, respectively.

Proof. Since $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a smooth morphism at x , replacing \mathfrak{X} , if necessary, by a smaller neighborhood of x , we may assume that $f : \mathfrak{X} = \mathrm{Spf}(A) \rightarrow \mathfrak{Y} = \mathrm{Spf}(B)$ is a morphism in $\mathrm{NFS}_{\mathrm{af}}$ smooth at x and that $\mathfrak{X}' = \mathrm{Spf}(A/I)$. Therefore, applying [AJP, Proposition 4.8] and Corollary 5.10 we have that $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1$ is a locally free $\mathcal{O}_{\mathfrak{X}}$ -Module of rank n .

Let us prove that (1) \Rightarrow (2). Replacing, again, if it is necessary \mathfrak{X}' by a smaller neighborhood of x , we may also assume that $f' : \mathfrak{X}' \rightarrow \mathfrak{Y}$ is a smooth morphism. Then, by an argument along the lines of the previous paragraph, there results that $\widehat{\Omega}_{\mathfrak{X}'/\mathfrak{Y}}^1$ is a locally free $\mathcal{O}_{\mathfrak{X}'}$ -Module of rank $n - m$. Zariski's Jacobian criterion for formal schemes ([AJP, Corollary 4.15]) implies that the sequence

$$0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}'} \rightarrow \widehat{\Omega}_{\mathfrak{X}'/\mathfrak{Y}}^1 \rightarrow 0$$

is exact and split, from where we deduce that $\mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_{\mathfrak{X}'}$ -Module of rank m .

Reciprocally, applying [EGA I, (0, 5.5.4)] to the Second Fundamental Exact Sequence ([AJP, Proposition 3.13]) associated to the morphisms $\mathfrak{X}' \xrightarrow{f'} \mathfrak{X} \xrightarrow{f} \mathfrak{X}$, we deduce that there exists an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}'$ with $x \in \mathfrak{U}$ such that

$$0 \rightarrow \left(\frac{\mathcal{I}}{\mathcal{I}^2} \right) |_{\mathfrak{U}} \rightarrow (\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}'}) |_{\mathfrak{U}} \rightarrow (\widehat{\Omega}_{\mathfrak{X}'/\mathfrak{Y}}^1) |_{\mathfrak{U}} \rightarrow 0$$

is exact and split. From Zariski's Jacobian criterion it follows that $f'|_{\mathfrak{U}}$ is smooth and therefore, f' is smooth at x . \square

Locally, a pseudo finite type morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ factors as $\mathfrak{U} \xrightarrow{j} \mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^r \xrightarrow{p} \mathfrak{Y}$ where j is a closed immersion (see [AJP, Proposition 1.7]). In Corollary 5.13 we provide a criterion in terms a matrix rank that tells whether \mathfrak{U} is smooth over \mathfrak{Y} or not.

²Let (X, \mathcal{O}_X) be a ringed space. We say that the sequence of \mathcal{O}_X -Modules $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ is exact at $x \in X$ if, and only if, $0 \rightarrow \mathcal{F}_x \rightarrow \mathcal{G}_x \rightarrow \mathcal{H}_x \rightarrow 0$ is an exact sequence of $\mathcal{O}_{X,x}$ -modules.

5.12. Given $\mathfrak{Y} = \mathrm{Spf}(A)$ in $\mathrm{NFS}_{\mathrm{af}}$ consider $\mathfrak{X} \subset \mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^s$ a closed formal subscheme given by an Ideal $\mathcal{I} = I^\Delta$ with $I = \langle g_1, g_2, \dots, g_k \rangle \subset A\{\mathbf{T}\}[[\mathbf{Z}]]$ being $\mathbf{T} = T_1, T_2, \dots, T_r$ and $\mathbf{Z} = Z_1, Z_2, \dots, Z_s$ two sets of of indeterminates. From [AJP, 3.14] we have that

$$\{\widehat{dT}_1, \dots, \widehat{dT}_r, \widehat{dZ}_1, \dots, \widehat{dZ}_s\}$$

is a basis of $\widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1$ and also that given $g \in A\{\mathbf{T}\}[[\mathbf{Z}]]$ it holds that:

$$\widehat{dg} = \sum_{i=1}^r \frac{\partial g}{\partial T_i} \widehat{dT}_i + \sum_{j=1}^s \frac{\partial g}{\partial Z_j} \widehat{dZ}_j$$

where \widehat{d} is the complete canonical derivation of $A\{\mathbf{T}\}[[\mathbf{Z}]]$ over A . For all $g \in A\{\mathbf{T}\}[[\mathbf{Z}]]$, $w \in \{\widehat{dT}_1, \dots, \widehat{dT}_r, \widehat{dZ}_1, \dots, \widehat{dZ}_s\}$ and $x \in \mathfrak{X}$, denote by $\frac{\partial g}{\partial w}(x)$ the image of $\frac{\partial g}{\partial w} \in A\{\mathbf{T}\}[[\mathbf{Z}]]$ in $k(x)$. We will call

$$\mathrm{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x) = \begin{pmatrix} \frac{\partial g_1}{\partial T_1}(x) & \dots & \frac{\partial g_1}{\partial T_r}(x) & \frac{\partial g_1}{\partial Z_1}(x) & \dots & \frac{\partial g_1}{\partial Z_s}(x) \\ \frac{\partial g_2}{\partial T_1}(x) & \dots & \frac{\partial g_2}{\partial T_r}(x) & \frac{\partial g_2}{\partial Z_1}(x) & \dots & \frac{\partial g_2}{\partial Z_s}(x) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial T_1}(x) & \dots & \frac{\partial g_k}{\partial T_r}(x) & \frac{\partial g_k}{\partial Z_1}(x) & \dots & \frac{\partial g_k}{\partial Z_s}(x) \end{pmatrix}.$$

the *Jacobian matrix of \mathfrak{X} over \mathfrak{Y} at x* . This matrix depends on the chosen generators of I and therefore, the notation $\mathrm{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x)$ is not completely accurate.

Corollary 5.13. (*Jacobian criterion for the affine formal space and the affine formal disc*). *With the previous notations, the following assertions are equivalent:*

- (1) *The morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is smooth at x and $\dim_x f = r + s - l$.*
- (2) *There exists a subset $\{g_1, g_2, \dots, g_l\} \subset \{g_1, g_2, \dots, g_k\}$ such that $\mathcal{I}_x = \langle g_1, g_2, \dots, g_l \rangle \mathcal{O}_{\mathfrak{X},x}$ and $\mathrm{rg}(\mathrm{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x)) = l$.*

Proof. Assume (1). By Proposition 5.11 we have that the sequence

$$0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \rightarrow \widehat{\Omega}_{\mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^s}^1 / \mathfrak{Y} \otimes_{\mathcal{O}_{\mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^s}} \mathcal{O}_{\mathfrak{X}} \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \rightarrow 0$$

is exact at x and the corresponding $\mathcal{O}_{\mathfrak{X}}$ -Modules are locally free, in a neighborhood of x , of ranks l , $r + s$ and $r + s - l$, respectively. Therefore,

$$(5.13.1) \quad 0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \otimes_{\mathcal{O}_{\mathfrak{X}}} k(x) \rightarrow \widehat{\Omega}_{\mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^s}^1 / \mathfrak{Y} \otimes_{\mathcal{O}_{\mathbb{D}_{\mathbb{A}_{\mathfrak{Y}}}^s}} k(x) \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} k(x) \rightarrow 0$$

is an exact sequence of $k(x)$ -vector spaces of dimension l , $r + s$, $r + s - l$, respectively. Thus, there exists a set $\{g_1, g_2, \dots, g_l\} \subset \{g_1, g_2, \dots, g_k\}$ such that $\{g_1(x), g_2(x), \dots, g_l(x)\}$ provides a basis of $\mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_{\mathfrak{X}}} k(x)$ at x . By Nakayama's lemma there results that $\mathcal{I}_x = \langle g_1, g_2, \dots, g_l \rangle \mathcal{O}_{\mathfrak{X},x}$. Besides,

from the exactness of the sequence (5.13.1) and from the equivalence of categories [EGA I, (10.10.2)] we deduce that the set

$$\{\widehat{dg}_1(x), \widehat{dg}_2(x), \dots, \widehat{dg}_l(x)\} \subset \widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1 \otimes_{A\{\mathbf{T}\}[[\mathbf{Z}]]} k(x)$$

is linearly independent. Therefore, $\text{rg}(\text{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x)) = l$.

Conversely, from the Second Fundamental Exact Sequence associated to the morphisms $\mathfrak{X} \hookrightarrow \mathbb{D}_{\mathbb{A}_y^s}^s \rightarrow \mathfrak{Y}$ [AJP, Proposition 3.13] we get the exact sequence

$$\frac{\mathcal{I}}{\mathcal{I}^2} \otimes_{\mathcal{O}_{\mathfrak{X}}} k(x) \rightarrow \widehat{\Omega}_{\mathbb{D}_{\mathbb{A}_y^s}^s/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathbb{D}_{\mathbb{A}_y^s}^s}} k(x) \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathfrak{X}}} k(x) \rightarrow 0.$$

Since $\text{rg}(\text{Jac}_{\mathfrak{X}/\mathfrak{Y}}(x)) = l$, we have that

$$\{\widehat{dg}_1(x), \widehat{dg}_2(x), \dots, \widehat{dg}_l(x)\} \subset \widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1 \otimes_{A\{\mathbf{T}\}[[\mathbf{Z}]]} k(x)$$

is a linearly independent set. Extending this set to a basis of the vector space $\widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1 \otimes_{A\{\mathbf{T}\}[[\mathbf{Z}]]} k(x)$, by Nakayama's lemma we find a basis $\mathcal{B} \subset \widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1$ such that $\{\widehat{dg}_1, \widehat{dg}_2, \dots, \widehat{dg}_l\} \subset \mathcal{B}$ and therefore

$$\{\widehat{dg}_1, \widehat{dg}_2, \dots, \widehat{dg}_l\} \subset \widehat{\Omega}_{A\{\mathbf{T}\}[[\mathbf{Z}]]/A}^1 \otimes_{A\{\mathbf{T}\}[[\mathbf{Z}]]} A\{\mathbf{T}\}[[\mathbf{Z}]]/I$$

is a linearly independent set at x . Thus the set $\{g_1, g_2, \dots, g_l\}$ provides a basis of $\mathcal{I}/\mathcal{I}^2$ at x and by the equivalence of categories [EGA I, (10.10.2)] we have that the sequence of $\mathcal{O}_{\mathfrak{X}}$ -Modules

$$0 \rightarrow \frac{\mathcal{I}}{\mathcal{I}^2} \rightarrow \widehat{\Omega}_{\mathbb{D}_{\mathbb{A}_y^s}^s/\mathfrak{Y}}^1 \otimes_{\mathcal{O}_{\mathbb{D}_{\mathbb{A}_y^s}^s}} \mathcal{O}_{\mathfrak{X}} \rightarrow \widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 \rightarrow 0$$

is split exact at x of locally free Modules of ranks l , $r + s$ and $r + s - l$, respectively. Applying Proposition 5.11 there results that f is smooth at x and $\dim_x f = r + s - l$. \square

Notice that the matrix form of the Jacobian criterion for the affine formal space and the affine formal disc (Corollary 5.13) generalize the usual matrix form of the Jacobian criterion for the affine space in Sch ([AIK1, Ch. VII, Theorem (5.14)]).

6. ÉTALE MORPHISMS

The main part of the results of this section are consequence of that obtained in Sections 4 and 5. These results will allow us to characterize in Section 7 two important classes of étale morphisms: open immersions and completion morphisms.

Proposition 6.1. *Given $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ in NFS let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. Using them, let us write*

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

If $f_n : X_n \rightarrow Y_n$ is étale, $\forall n \in \mathbb{N}$, then f is étale.

Proof. The sum of Proposition 4.1 and Proposition 5.1. \square

Corollary 6.2. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an adic morphism in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition. Consider $\{f_n\}_{n \in \mathbb{N}}$ the direct system of morphisms of schemes associated to the Ideals of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$. The morphism f is étale if, and only if, the morphisms $f_n : X_n \rightarrow Y_n$ are étale $\forall n \in \mathbb{N}$.*

Proof. It follows from Proposition 4.1 and Corollary 5.2. \square

Proposition 6.3. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an adic morphism in NFS and let $f_0 : X_0 \rightarrow Y_0$ be the morphism of schemes associated to the Ideals of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$. Then, f is étale if, and only if, f is flat and f_0 is étale.*

Proof. Put together Proposition 4.3 and Corollary 5.6. \square

Note that example 5.3 on page 24 shows that in the non adic case the last two results do not hold and also that, in general, the converse of Proposition 6.1 is not true.

Proposition 6.4. *Let f be a pseudo finite type morphism in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and, with this choice, let us write*

$$f = \varinjlim_{n \in \mathbb{N}} f_n.$$

If $f_0 : X_0 \rightarrow Y_0$ is étale, f is flat and $f^{-1}(f(x)) = f_0^{-1}(f(x))$, for all $x \in \mathfrak{X}$, then f is étale.

Proof. It follows from Corollary 4.10 and Corollary 5.8. \square

Example 5.3 shows that the reciprocal of the last result is not true. Next Proposition gives us a local characterization of étale morphisms.

Proposition 6.5. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS of pseudo finite type, let $x \in \mathfrak{X}$ and $y = f(x)$, the following conditions are equivalent:*

- (1) f is étale at x .
- (2) $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally étale $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ -algebra for the adic topologies.
- (2') $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally étale $\widehat{\mathcal{O}_{\mathfrak{Y},y}}$ -algebra for the adic topologies.
- (3) f is flat at x and $f^{-1}(y)$ is a $k(y)$ -formal scheme étale at x .
- (4) f is flat and unramified at x .
- (4') f is flat at x and $(\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1)_x = 0$.
- (5) f is smooth at x and a quasi-covering at x .

Proof. Applying Proposition 4.5 and Proposition 5.4 we have that

$$(5) \Leftrightarrow (1) \Leftrightarrow (2) \Leftrightarrow (2') \Leftrightarrow (3) \Rightarrow (4) \Leftrightarrow (4').$$

Let $C := \widehat{\mathcal{O}_{\mathfrak{X},x}} \otimes_{\widehat{\mathcal{O}_{\mathfrak{Y},y}}} k(y)$. To show (4) \Rightarrow (5), by Corollary 4.7 it is only left to prove that f is smooth at x . By hypothesis, we have that f is unramified

at x and by Proposition 4.5, there results that $C = k(x)$ and $k(x)|k(y)$ is a finite separable extension, therefore, formally étale. Since f is flat at x , by Proposition 5.4 we conclude that f is smooth at x .

To prove that (5) \Rightarrow (1), it suffices to check that f is unramified at x or, equivalently by Proposition 4.5, that $C = k(x)$ and that $k(x)|k(y)$ is a finite separable extension. As f is smooth at x , applying Proposition 5.4, we have that $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally smooth $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ -algebra for the adic topologies. Then by base-change there results that C is a formally smooth $k(y)$ -algebra. By [EGA IV₁, (0, 19.3.8)] we have that C is a formally smooth $k(y)$ -algebra for the topologies given by the maximal ideals and from [M2, Lemma 1, p. 216] it holds that C is a regular local ring. Besides, by hypothesis we have that C is a finite $k(y)$ -module, therefore, an artinian ring, so $C = k(x)$. Since $k(x) = C$ is a formally smooth $k(y)$ -algebra we have that $k(x)|k(y)$ is a separable extension (*cf.* [EGA IV₁, (0, 19.6.1)]). \square

Corollary 6.6. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. The following conditions are equivalent:*

- (1) f is étale.
- (2) For all $x \in \mathfrak{X}$, $\mathcal{O}_{\mathfrak{X},x}$ is a formally étale $\mathcal{O}_{\mathfrak{Y},f(x)}$ -algebra for the adic topologies.
- (2') For all $x \in \mathfrak{X}$, $\widehat{\mathcal{O}_{\mathfrak{X},x}}$ is a formally étale $\widehat{\mathcal{O}_{\mathfrak{Y},f(x)}}$ -algebra for the adic topologies.
- (3) For all $x \in \mathfrak{X}$, $f^{-1}(f(x))$ is a $k(f(x))$ -formal scheme étale at x and f is flat.
- (4) f is flat and unramified.
- (4') f is flat and $\widehat{\Omega}_{\mathfrak{X}/\mathfrak{Y}}^1 = 0$.
- (5) f is smooth and a quasi-covering.

Example 6.7. Given a field K , the canonical morphism of projection $\mathbb{D}_K^1 \rightarrow \text{Spec}(K)$ is smooth, pseudo quasi-finite but it is not étale.

In Sch a morphism is étale if, and only if, it is smooth and quasi-finite. The previous example shows that in NFS there are smooth and pseudo quasi-finite morphisms that are not étale. That is why we consider quasi-coverings in NFS (see Definition 2.8) as the right generalization of quasi-finite morphisms in Sch.

7. STRUCTURE THEOREMS OF THE INFINITESIMAL LIFTING PROPERTIES

We begin with two results that will be used in the proof of the remainder results of this section.

Proposition 7.1. *In NFS let us consider a formally étale morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$, a morphism $g : \mathfrak{S} \rightarrow \mathfrak{Y}$ and $\mathcal{L} \subset \mathcal{O}_{\mathfrak{S}}$ an Ideal of definition of \mathfrak{S} . Let us write with respect to \mathcal{L}*

$$\mathfrak{S} = \varinjlim_{n \in \mathbb{N}} S_n.$$

If $h_0 : S_0 \rightarrow \mathfrak{X}$ is a morphism in NFS that makes the diagram

$$\begin{array}{ccc} S_0 & \hookrightarrow & \mathfrak{S} \\ h_0 \downarrow & & \downarrow g \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

commutative, where $S_0 \hookrightarrow \mathfrak{S}$ is the canonical closed immersion, then there exists a unique \mathfrak{Y} -morphism $l : \mathfrak{S} \rightarrow \mathfrak{X}$ in NFS such that $l|_{S_0} = h_0$.

Proof. By induction on n we are going to construct a collection of morphisms $\{h_n : S_n \rightarrow \mathfrak{X}\}_{n \in \mathbb{N}}$ such that the diagrams

$$\begin{array}{ccccc} & & S_{n-1} & & \\ & & \searrow & & \searrow \\ & & & S_n & \hookrightarrow & \mathfrak{S} \\ & h_{n-1} \searrow & & \downarrow h_n & & \downarrow g \\ & & & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

commute. For $n = 1$, by [AJP, 2.4] there exists a unique morphism $h_1 : S_1 \rightarrow \mathfrak{X}$ such that $h_1|_{S_0} = h_0$ and $g|_{S_1} = f \circ h_1$. Let now $n \in \mathbb{N}$, if we already have for all $0 < k < n$ morphisms $h_k : S_k \rightarrow \mathfrak{X}$ such that $h_k|_{S_{k-1}} = h_{k-1}$ and $g|_{S_k} = f \circ h_k$, by [AJP, *loc. cit.*] there exists an unique morphism $h_n : S_n \rightarrow \mathfrak{X}$ such that $h_n|_{S_{n-1}} = h_{n-1}$ and $g|_{S_n} = f \circ h_n$. It is straightforward that

$$l := \varinjlim_{n \in \mathbb{N}} h_n$$

is a morphism of formal schemes and is the unique such that the diagram

$$\begin{array}{ccc} S_0 & \hookrightarrow & \mathfrak{S} \\ h_0 \downarrow & \swarrow h & \downarrow g \\ \mathfrak{X} & \longrightarrow & \mathfrak{Y} \end{array}$$

commutes. □

Corollary 7.2. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an étale morphism in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ such that the corresponding morphism $f_0 : X_0 \rightarrow Y_0$ is an isomorphism. Then f is an isomorphism.*

Proof. By Proposition 7.1 there exists a unique morphism $g : \mathfrak{Y} \rightarrow \mathfrak{X}$ such that the following diagram is commutative

$$\begin{array}{ccc}
 Y_0 & \hookrightarrow & \mathfrak{Y} \\
 f_0^{-1} \downarrow & & \downarrow 1_{\mathfrak{Y}} \\
 X_0 & \xrightarrow{g} & \mathfrak{Y} \\
 \downarrow \wr & & \downarrow \\
 \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y}
 \end{array}$$

Then, by [AJP, Proposition 2.13] it follows that g is an étale morphism. Thus, applying Proposition 7.1 we have that there exists a unique morphism $f' : \mathfrak{X} \rightarrow \mathfrak{Y}$ such that the following diagram is commutative

$$\begin{array}{ccc}
 X_0 & \hookrightarrow & \mathfrak{X} \\
 f_0 \downarrow & & \downarrow 1_{\mathfrak{X}} \\
 Y_0 & \xrightarrow{f'} & \mathfrak{X} \\
 \downarrow \wr & & \downarrow \\
 \mathfrak{Y} & \xrightarrow{g} & \mathfrak{X}
 \end{array}$$

Necessarily $f = f'$ and f is an isomorphism. \square

In Sch open immersions are characterized as being those étale morphisms that are radical (see [EGA IV₄, (17.9.1)]). In the following theorem we extend this characterization and relate open immersions in formal schemes with their counterparts in schemes.

Theorem 7.3. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS. The following conditions are equivalent:*

- (1) f is an open immersion.
- (2) f is adic, flat and if $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition such that $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$, the associated morphism of schemes $f_0 : X_0 \rightarrow Y_0$ is an open immersion.
- (3) f is adic étale and radical.
- (4) There are $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition satisfying that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ such that the morphisms $f_n : X_n \rightarrow Y_n$ are open immersions, for all $n \in \mathbb{N}$.

Proof. The implication (1) \Rightarrow (2) is immediate. Given $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition, assume (2) and let us show (3). Since f_0 is an open immersion, is radical, so, f is radical (see Definition 2.5 and its attached paragraph). Furthermore, f is flat and f_0 is an étale morphism then f is étale (see Proposition 6.3). Let us prove that (3) \Rightarrow (4). Given $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$, by Corollary 6.2 the morphisms $f_n : X_n \rightarrow Y_n$ are étale, for all $n \in \mathbb{N}$. The morphisms f_n are also radical for all $n \in \mathbb{N}$

(see Definition 2.5) and thus by [EGA IV₄, (17.9.1)] it follows that f_n is an open immersion, for all $n \in \mathbb{N}$. Finally, suppose that (4) holds and let us see that f is an open immersion. With the notations of (4), there exists an open subset $U_0 \subset Y_0$ such that f_0 factors as

$$X_0 \xrightarrow{f'_0} U_0 \xrightarrow{i_0} Y_0$$

where f'_0 is an isomorphism and i_0 is the canonical inclusion. Let $\mathfrak{U} \subset \mathfrak{Y}$ be the open formal subscheme with underlying topological space U_0 . Since the open immersion $i : \mathfrak{U} \rightarrow \mathfrak{Y}$ is étale, then Proposition 7.1 implies that there exists a morphism $f' : \mathfrak{X} \rightarrow \mathfrak{U}$ of formal schemes such that the diagram:

$$\begin{array}{ccccc}
 \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & & \\
 & \searrow f' & \mathfrak{U} & \xrightarrow{i} & \mathfrak{Y} \\
 & & \uparrow & & \uparrow \\
 X_0 & \xrightarrow{f_0} & Y_0 & & Y_0 \\
 & \searrow f'_0 & U_0 & \xrightarrow{i_0} & Y_0 \\
 & & \uparrow & & \uparrow
 \end{array}$$

is commutative. Since the morphisms f_n are étale, for all $n \in \mathbb{N}$, Proposition 6.1 implies that f is étale. By [AJP, Proposition 2.13] we have that f' is étale and applying Corollary 7.2, f' is an isomorphism and therefore, f is an open immersion. \square

Corollary 7.4. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a pseudo finite type morphism in NFS. Then f is unramified if, and only if, the diagonal morphism $\Delta_f : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an open embedding.*

Proof. Take $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and the map f can be expressed as the limit of maps of usual schemes $f_n : X_n \rightarrow Y_n$. The morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is unramified if, and only if, f_n is unramified for all $n \in \mathbb{N}$ by Proposition 4.1. By [EGA IV₄, Corollaire (17.4.2)] this is equivalent to $\Delta_{f_n} : X_n \rightarrow X_n \times_{Y_n} X_n$ being an open embedding for all $n \in \mathbb{N}$. But this, in turn, is equivalent to the fact that $\Delta_f : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an open embedding by Theorem 7.3. \square

Every completion morphism is a pseudo closed immersion that is flat (*cf.* Proposition 3.7). Next, we prove that this condition is also sufficient. Thus, we obtain a criterion to determine whether a \mathfrak{Y} -formal scheme \mathfrak{X} is the completion of \mathfrak{Y} along a closed formal subscheme.

Theorem 7.5. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS and let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. Let us write $f_0 : X_0 \rightarrow Y_0$ the corresponding morphism of ordinary schemes. The following conditions are equivalent:*

- (1) *There exists a closed formal subscheme $\mathfrak{Y}' \subset \mathfrak{Y}$ such that $\mathfrak{X} = \mathfrak{Y}/_{\mathfrak{Y}'}$ and f is the morphism of completion of \mathfrak{Y} along \mathfrak{Y}' .*
- (2) *The morphism f is a flat pseudo closed immersion.*
- (3) *The morphism f is étale and $f_0 : X_0 \rightarrow Y_0$ is a closed immersion.*
- (4) *The morphism f is a smooth pseudo closed immersion.*

Proof. The implication (1) \Rightarrow (2) is Proposition 3.7. Let us show that (2) \Rightarrow (3). Since f is a pseudo closed immersion, by Corollary 4.13 we have that f is unramified. Then as f is flat, Corollary 6.6 establishes that f is étale. The equivalence (3) \Leftrightarrow (4) is consequence of Corollary 4.13. Finally, we show that (3) \Rightarrow (1). By hypothesis, the morphism $f_0 : X_0 \rightarrow Y_0$ is a closed immersion. Consider $\kappa : \mathfrak{Y}/_{X_0} \rightarrow \mathfrak{Y}$ the morphism of completion of \mathfrak{Y} along X_0 and let us prove that \mathfrak{X} and $\mathfrak{Y}/_{X_0}$ are \mathfrak{Y} -isomorphic. By Proposition 3.7 the morphism κ is étale so, applying Proposition 7.1, we have that there exists a \mathfrak{Y} -morphism $\varphi : \mathfrak{X} \rightarrow \mathfrak{Y}/_{X_0}$ such that the following diagram is commutative

$$\begin{array}{ccccc}
 \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & & \\
 & \searrow \varphi & \nearrow \kappa & & \\
 & & \mathfrak{Y}/_{X_0} & & \\
 \uparrow & & \uparrow & & \uparrow \\
 X_0 & \xrightarrow{f_0} & Y_0 & & \\
 & \searrow \varphi_0=1_{X_0} & \nearrow f_0 & & \\
 & & X_0 & &
 \end{array}$$

From [AJP, Proposition 2.13] there results that φ is étale and then by Corollary 7.2 we get that φ is an isomorphism. \square

Remark. A consequence of the proof of (3) \Rightarrow (1) is the following: Given \mathfrak{Y} in NFS and a closed formal subscheme $\mathfrak{Y}' \subset \mathfrak{Y}$ defined by the Ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Y}}$, then for every Ideal of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ of \mathfrak{Y} , it holds that

$$\mathfrak{Y}/_{\mathfrak{Y}'} = \mathfrak{Y}/_{Y'_0}$$

where $Y'_0 = (\mathfrak{Y}', \mathcal{O}_{\mathfrak{Y}}/(\mathcal{I} + \mathcal{K}))$.

7.6. Given a scheme Y and a closed subscheme $Y_0 \subset Y$ with the same topological space, the functor $X \rightsquigarrow X \times_Y Y_0$ defines an equivalence between the category of étale Y -schemes and the category of étale Y_0 -schemes by [EGA IV₄, (18.1.2)]. In the next theorem we extend this equivalence to the category of locally noetherian formal schemes. A special case of this theorem, namely when \mathfrak{Y} is smooth over a noetherian ordinary base scheme, appears in [Y, Proposition 2.4].

Proposition 7.7. *Let \mathfrak{Y} be in NFS and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition such that*

$$\mathfrak{Y} = \varinjlim_{n \in \mathbb{N}} Y_n.$$

Then the functor

$$\begin{array}{ccc} \text{étale adic } \mathfrak{Y}\text{-formal schemes} & \xrightarrow{F} & \text{étale } Y_0\text{-schemes} \\ \mathfrak{X} & \rightsquigarrow & \mathfrak{X} \times_{\mathfrak{Y}} Y_0 \end{array}$$

is an equivalence of categories.

Proof. By [McL, IV, §4, Theorem 1] it suffices to prove that: (a) F is full and faithful; (b) Given X_0 an étale Y_0 -scheme there exists an étale adic \mathfrak{Y} -formal scheme \mathfrak{X} such that $F(\mathfrak{X}) = \mathfrak{X} \times_{\mathfrak{Y}} Y_0 \cong X_0$.

The assertion (a) is an immediate consequence of Proposition 7.1.

Let us show (b). Given X_0 an étale Y_0 -scheme in Sch by [EGA IV₄, (18.1.2)] there exists X_1 a locally noetherian étale Y_1 -scheme such that $X_1 \times_{Y_1} Y_0 \cong X_0$. Reasoning by induction on $n \in \mathbb{N}$ and using [EGA IV₄, *loc. cit.*], we get a family $\{X_n\}_{n \in \mathbb{N}}$ such that, for all $n \in \mathbb{N}$, X_n is a locally noetherian étale Y_n -scheme and $X_n \times_{Y_n} Y_{n-1} \cong X_{n-1}$, for $n > 0$. Then

$$\mathfrak{X} := \varinjlim_{n \in \mathbb{N}} X_n$$

is a locally noetherian adic \mathfrak{Y} -formal scheme (by [EGA I, (10.12.3.1)]),

$$\mathfrak{X} \times_{\mathfrak{Y}} Y_0 \underset{[\text{EGA I, (10.7.4)}]}{=} \varinjlim_{n \in \mathbb{N}} (X_n \times_{Y_n} Y_0) = X_0$$

and \mathfrak{X} is an étale \mathfrak{Y} -formal scheme (see Proposition 6.1). \square

Remark. It seems plausible that there is a theory of an algebraic fundamental group for formal schemes that classifies *adic* étale surjective maps onto a noetherian formal scheme \mathfrak{X} . If this is the case, the previous theorem would imply that it agrees with the fundamental group of X_0 . We also consider feasible the existence of a bigger fundamental group classifying arbitrary étale surjective maps onto a noetherian formal scheme \mathfrak{X} , that would give additional information about \mathfrak{X} .

Corollary 7.8. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an étale morphism in NFS. Given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$, and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, if the induced morphism $f_0 : X_0 \rightarrow Y_0$ is étale, then f is adic étale.*

Proof. By Proposition 7.7 there is an adic étale morphism $f' : \mathfrak{X}' \rightarrow \mathfrak{Y}$ in NFS such that $\mathfrak{X}' \times_{\mathfrak{Y}} Y_0 = X_0$. Therefore by Proposition 7.1 there exists a morphism of formal schemes $g : \mathfrak{X} \rightarrow \mathfrak{X}'$ such that the diagram

$$\begin{array}{ccccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} & & \\ & \searrow g & \nearrow f' & & \\ & & \mathfrak{X}' & & \\ \uparrow & & \uparrow & & \uparrow \\ X_0 & \xrightarrow{\quad} & X_0 & \xrightarrow{f_0} & Y_0 \\ & \searrow g_0=1_{X_0} & \nearrow f'_0 & & \\ & & X_0 & & \end{array}$$

is commutative. Applying [AJP, Proposition 2.13] we have that g is étale and from Corollary 7.2 we deduce that g is an isomorphism and therefore, f is adic étale. \square

Corollary 7.9. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS. The morphism f is adic étale if, and only if, there exist $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ such that the induced morphisms $f_n : X_n \rightarrow Y_n$ are étale, for all $n \in \mathbb{N}$.*

Proof. If f is adic étale, given $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ an Ideal of definition, take $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$ the corresponding Ideal of definition of \mathfrak{X} . By base change, we have that the morphisms $f_n : X_n \rightarrow Y_n$ are étale, for all $n \in \mathbb{N}$. The reciprocal is a consequence of Proposition 6.1 and of the previous Corollary. \square

Proposition 7.7 says that given

$$\mathfrak{Y} = \varinjlim_{n \in \mathbb{N}} Y_n$$

in NFS and X_0 an étale Y_0 -scheme there exists a unique (up to isomorphism) étale \mathfrak{Y} -formal scheme \mathfrak{X} such that $\mathfrak{X} \times_{\mathfrak{Y}} Y_0 = X_0$. But, what happens when X_0 is a smooth Y_0 -scheme?

Proposition 7.10. *Let \mathfrak{Y} be in NFS and with respect to an Ideal of definition $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ let us write*

$$\mathfrak{Y} = \varinjlim_{n \in \mathbb{N}} Y_n.$$

Given $f_0 : X_0 \rightarrow Y_0$ a morphism in Sch smooth at $x \in X_0$, there exists an open subset $U_0 \subset X_0$, with $x \in U_0$ and a smooth adic \mathfrak{Y} -formal scheme \mathfrak{U} such that $\mathfrak{U} \times_{\mathfrak{Y}} Y_0 \cong U_0$.

Proof. Since this is a local question in \mathfrak{Y} , we may assume that $\mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathrm{NFS}_{\mathrm{af}}$, $\mathcal{K} = K^{\Delta}$ with $K \subset B$ an ideal of definition of the adic ring B , $B_0 = B/K$ and $f_0 : X_0 = \mathrm{Spec}(A_0) \rightarrow Y_0 = \mathrm{Spec}(B_0)$ is a morphism in $\mathrm{Sch}_{\mathrm{af}}$ smooth at $x \in X_0$. By Proposition 5.9 there exists an open subset $U_0 \subset X_0$ with $x \in U_0$ such that $f_0|_{U_0}$ factors as

$$U_0 \xrightarrow{f'_0} \mathbb{A}_{Y_0}^n = \mathrm{Spec}(B_0[\mathbf{T}]) \xrightarrow{p_0} Y_0$$

where f'_0 is an étale morphism and p_0 is the canonical projection, being $\mathbf{T} = T_1, T_2, \dots, T_r$ a set of indeterminates. The morphism p_0 lifts to a morphism of projection $p : \mathbb{A}_{\mathfrak{Y}}^n = \mathrm{Spf}(B\{\mathbf{T}\}) \rightarrow \mathfrak{Y}$ such that the following diagram is cartesian

$$\begin{array}{ccc} \mathbb{A}_{\mathfrak{Y}}^n & \xrightarrow{p} & \mathfrak{Y} \\ \uparrow & & \uparrow \\ U_0 & \xrightarrow{f'_0} \mathbb{A}_{Y_0}^n \xrightarrow{p_0} & Y_0 \end{array}$$

Applying Proposition 7.7, there exists a locally noetherian étale adic $\mathbb{A}_{\mathfrak{Y}}^n$ -formal scheme \mathfrak{U} such that $U_0 \cong \mathfrak{U} \times_{\mathbb{A}_{\mathfrak{Y}}^n} \mathbb{A}_{Y_0}^n$. Then \mathfrak{U} is a smooth adic \mathfrak{Y} -formal scheme such that $U_0 \cong \mathfrak{U} \times_{\mathfrak{Y}} Y_0$. \square

The next theorem transfers the local description of unramified morphisms known in the case of schemes ([EGA IV₄, (18.4.7)]) to the framework of formal schemes.

Theorem 7.11. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS unramified at $x \in \mathfrak{X}$. Then there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}$$

where κ is a pseudo closed immersion and f' is an adic étale morphism.

Proof. Let $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. The morphism of schemes f_0 associated to these Ideals is unramified at x (Proposition 4.1) and by [EGA IV₄, (18.4.7)] there exists an open set $U_0 \subset X_0$ with $x \in U_0$ such that $f_0|_{U_0}$ factors as

$$U_0 \xrightarrow{\kappa_0} X'_0 \xrightarrow{f'_0} Y_0$$

where κ_0 is a closed immersion and f'_0 is an étale morphism. Proposition 7.7 implies that there exists an étale adic morphism $f' : \mathfrak{X}' \rightarrow \mathfrak{Y}$ in NFS such that $\mathfrak{X}' \times_{\mathfrak{Y}} Y_0 = X'_0$. Then if $\mathfrak{U} \subset \mathfrak{X}$ is the open formal scheme with underlying topological space U_0 , by Proposition 7.1 there exists a morphism $\kappa : \mathfrak{U} \rightarrow \mathfrak{X}'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} \mathfrak{U} & \xrightarrow{f|_{\mathfrak{U}}} & & \mathfrak{Y} & \\ & \searrow \kappa & & \nearrow f' & \\ & & \mathfrak{X}' & & \\ \uparrow \cup & & \uparrow & & \downarrow \cup \\ U_0 & \xrightarrow{f_0|_{U_0}} & & Y_0 & \\ & \searrow \kappa_0 & & \nearrow f'_0 & \\ & & X'_0 & & \end{array}$$

Since f is unramified, by [AJP, Proposition 2.13] it holds that κ is unramified. Furthermore, κ_0 is a closed immersion, then Corollary 4.13 shows us that κ is a pseudo closed immersion. \square

As a consequence of the last result we obtain the following local description for étale morphisms.

Theorem 7.12. *Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS étale at $x \in \mathfrak{X}$. Then there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}$$

where κ is a completion morphism and f' is an adic étale morphism.

Proof. By the last theorem we have that there exists an open formal subscheme $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as

$$\mathfrak{U} \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}$$

where κ is a pseudo closed immersion and f' is an adic étale morphism. Then since $f|_{\mathfrak{U}}$ is étale and f' is an adic étale morphism, by [AJP, Proposition 2.13] we have that κ is étale and applying Theorem 7.5 there results that κ is a completion morphism. \square

Theorem 7.13. *Let $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism in NFS smooth at $x \in \mathfrak{X}$. Then there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f'} \mathfrak{Y}$$

where κ is a completion morphism and f' is an adic smooth morphism.

Proof. By Proposition 5.9 there exists an open formal subscheme $\mathfrak{V} \subset \mathfrak{X}$ with $x \in \mathfrak{V}$ such that $f|_{\mathfrak{V}}$ factors as

$$\mathfrak{V} \xrightarrow{g} \mathbb{A}_{\mathfrak{Y}}^n \xrightarrow{p} \mathfrak{Y}$$

where g is étale and p is the canonical projection. Applying the last Theorem to the morphism g we conclude that there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ factors as

$$\mathfrak{U} \xrightarrow{\kappa} \mathfrak{X}' \xrightarrow{f''} \mathbb{A}_{\mathfrak{Y}}^n \xrightarrow{p} \mathfrak{Y}$$

where κ is a completion morphism, f'' is an adic étale morphism and p is the canonical projection, from where it follows that $f' = f'' \circ p$ is adic smooth. \square

Remark. Lipman, Nayak and Sastry note in [LNS, pag. 132] that this Theorem may simplify some developemts related to Cousin complexes and duality on formal schemes. See the final part of Remark 10.3.10 of *loc. cit.*

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