DIVISORS AND INVERTIBLE SHEAVES ON NOETHERIAN SCHEMES

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ABSTRACT. Given X a noetherian scheme, the canonical map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$, between the class group of Cartier divisors of X and the Picard group is not an isomorphism in general. In this note, we show that if X is a projective scheme over a noetherian ring A, this map is an isomorphism. We also give an example of a non-projective complete scheme over a field k on which exists an invertible sheaf that is not associated to any Cartier divisor.

INTRODUCTION

The divisors are a global invariant that play a very important role in the study of the geometry on an algebraic variety. It's well-known that there is an isomorphism between the Cartier divisor class group and the Picard group on a variety. It's also known that given a divisor on a scheme it has an invertible sheaf associated. The aim of this talk is to prove that on a (non necesssarily reduced) projective scheme X over a noetherian ring the Cartier divisor class group, $\mathrm{ClCa}(X)$, agree with the Picard group, $\mathrm{Pic}(X)$. We're guided by the idea given by Nakai in [N]. Nakai proved that on a projective scheme X over a field, the groups $\mathrm{ClCa}(X)$ and $\mathrm{Pic}(X)$ are isomorphic. His results are correct but some of his arguments are not completely right.

We also give an unpublished counterexample due to Kleiman of a complete, non-reduced and non-projective scheme over an algebraically closed field in which there's an invertible sheaf that is not associated to any Cartier divisor.

1. DEFINITIONS

Let $(X, \mathcal{O}_X)$ be a variety (a variety is a topological space X with a sheaf of k-funtions $\mathcal{O}_X$ such that X is locally isomorphic to an irreducible algebraic set). The ring of rational funtions, $\mathrm{Rat}(X)$, defines a locally constant sheaf on X that we'll denote $\mathcal{K}_X$.

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Definition. A *Cartier divisor* is a global section D of the sheaf $\mathcal{K}_X^*/\mathcal{O}_X^*$, that is, D is given by an open covering of X, $\{U_i\}_{i\in I}$ and, for all $i \in I$, sections $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$, such that for all $i, j \in I$, $\frac{f_i}{f_j} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.

The set of Cartier divisors is denoted by $\mathrm{Div_C}(X)$ and it's a group with the following operation: given $D_1 = \{(U_i, f_i)\}$ and $D_2 = \{(U_i, g_i)\}$ two divisors, we define D_1+D_2 as the divisor represented by $\{(U_i, f_i g_i)\}$.

A rational funtion $f \in \Gamma(X, \mathcal{K}_X^*)$ defines a *principal Cartier divisor* through the canonic homomorphism

$$\mathrm{div}_C : \Gamma(X, \mathcal{K}_X^*) \longrightarrow \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*).$$

We denote a principal divisor by $\mathrm{div}_C(f)$.

The group of Cartier divisors module the image of div_C is called the *Cartier divisor class group* that we write $\mathrm{ClCa}(X)$.

There is another global invariant on a variety, the Picard group, and is defined on a ringed space $(X, \mathcal{O}_X)$ in general.

Definition. The *Picard group* of X, $\mathrm{Pic}(X)$, is the set of isomorphism classes of invertible sheaves on X with the operation $\otimes$. The Picard group can be interpreted in language of cohomology by the next theorem.

Theorem 1.1. [EGA I, (5.6.3)] *Let $(X, \mathcal{O}_X)$ be a ringed space. Then*

$$\mathrm{Pic}(X) \approxeq H^1(X, \mathcal{O}_X^*).$$

It's well-known there is a relation between the Cartier divisor class group and the Picard group on a *variety* X: given a Cartier divisor D represented by $\{(U_i, f_i)\}$ we define the subsheaf $\mathcal{O}_X(D)$ of $\mathcal{K}_X$ as the submodule of $\mathcal{K}_X$ generated by $\{f_i^{-1}\}$ in $\{U_i\}$. Moreover, if D is a principal divisor $\mathcal{O}_X(D) \approxeq \mathcal{O}_X$. Then there is a monomorphism of groups $\mathrm{ClCa}(X) \longrightarrow \mathrm{Pic}(X)$ which is in fact an isomorphism.

And, what happens on general schemes? Let $(X, \mathcal{O}_X)$ be a scheme. Since $\mathrm{Rat}(X)$ is not defined on a non-reduced scheme it's necessary to extend this concept.

Definition. The *sheaf of total quotient rings*, $\mathcal{K}_X$, is the sheaf associated to the presheaf

$$U \longrightarrow \mathcal{K}_X^{\mathrm{p}}(U) = S(U)^{-1}\Gamma(U, \mathcal{O}_X)$$

where $S(U) = \{s \in \Gamma(U, \mathcal{O}_X) \,/\, s_p \text{ is a non zero divisor in } \mathcal{O}_{X,p} \, \forall p \in U\}$.

If $(X, \mathcal{O}_X)$ is an integral scheme this sheaf is the locally constant sheaf given by $\mathrm{Rat}(X)$.

This sheaf is not defined correctly in several standard references, however a correct treatment is Kleiman's article [K].

With the sheaf $\mathcal{K}_X$ we define the Cartier divisor class group of $(X, \mathcal{O}_X)$ in the same way that we've done for varieties. We also have a monomorphism of groups $\mathrm{ClCa}(X) \longrightarrow \mathrm{Pic}(X)$ which is an isomorphism if $(X, \mathcal{O}_X)$ is integral.

2. Divisors and invertible sheaves on projective schemes

Let X be a projective scheme with coordinates ring $S = \bigoplus_{i \in \mathbb{N}} S_i$ and $S_0 = A$ a noetherian ring.

The main result of this paper is

Theorem 2.1. *If X is a projective scheme over a noetherian ring A, then $\mathrm{ClCa}(X)$ and $\mathrm{Pic}(X)$ are naturally isomorphic.*

We're going to give an explicit argument using a ring of homogenous coordinates of X. Nakai's key argument is the election of a ring of homogenous coordinates with all of its generators non zero divisors. However, in this talk is proved that is sufficient to find a non irrelevant ring.

Definition. A graduate ring S is *irrelevant* if any element of $S_+ = \oplus_{i>0} S_i$ is a zero divisor of S.

Proposition 2.2. *Let X be a projective scheme over a noetherian ring A. Then it's possible to find a non-irrelevant ring of homogenous coordinates for X.*

Corolary 2.3. *Let X be a projective scheme over a noetherian ring A. Then it's possible to find an homogenous coordinate ring for X with a non zero divisor homogenous of degree one.*

Definition. Let X be a locally noetherian scheme. A open subset U of X is *schematically dense* if $\mathrm{Ass}(X) \subset U$ (*cfr.* [EGA IV$_4$, (11.10.2)]).

If X is a projective scheme and U is an open set of the form $D_+(f)$, U is schematically dense if f is a non zero divisor.

Proposition 2.4. *Let X be a projective scheme over a noetherian ring A and U a schematically dense open set of the form $D_+(f)$ where f is an homogenous element of degree 1 non zero divisor. Then the canonic homomorphism*

$$\Gamma(D_+(g), \mathcal{K}_X) \longrightarrow \Gamma(D_+(f) \cap D_+(g), \mathcal{K}_X)$$

is an isomorphism, for all homogenous elements $g \in S_+$.

Proposition 2.5. *Given B a noetherian ring, $X = \operatorname{Spec}(B)$, M a finitely generated B-module, and*

$$\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n \in \operatorname{Spec}(B)$$

comaximal ideals such that $M \otimes B_{\mathfrak{p}_i}$ is a rank-1 free $B_{\mathfrak{p}_i}$-module , $\forall i = 1, \ldots, n$. Then, there is a open set U of the form $D_+(g)$ with $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in U$ and the quasi-coherent $\mathcal{O}_X$-module $\mathcal{M} = \widetilde{M}$ is such that $\mathcal{M}|_U$ is a rank-1 free $\mathcal{O}_U$-Module .

Proof of the Theorem 2.6. By Corollary 2.3 it's possible to find a coordinate ring S for X such that there exits an homogenous element $y \in S_+$ of degree one non zero divisor. Then the open set $D_+(y)$ is schematically dense.

Let $\mathcal{L}$ be an invertible sheaf on X. The sheaf $\mathcal{L}$ is coherent and, then $\mathcal{L}|_{D_+(y)}$ is also a coherent sheaf. If we call $M = \Gamma(D_+(y), \mathcal{L})$, therefore $\mathcal{L}|_{D_+(y)} \approxeq \widetilde{M}$. Moreover, M is a finitely generated $S_{(y)}$-module.

On the other hand, by Proposition 2.5 given $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n$ the maximal ideals of the set $\operatorname{Ass}(S)$ and $p_1, \ldots, p_n \in \operatorname{Ass}(X)$ the corresponding points of X, there is an affine open set U_1 of $D_+(y)$ such that $\operatorname{Ass}(X) \subset U_1$. Moreover there exists an element $f_1 \in M$ with

$$\mathcal{L}_p = M \otimes \mathcal{O}_{X,p} = \mathcal{O}_{X,p}(f_1 \otimes 1) \qquad (\forall p \in U_1).$$

Let $s \in S$ a homogenous element such that $U_1 = D_+(s)$. Since $U_1 \subset X$ is schematically dense, s has to be a non zero divisor of S. Let

$$\mathcal{L}|_{U_1} \xrightarrow{\approxeq} \mathcal{O}_X|_{U_1}$$

the isomorphism determinated by the section $f_1 \otimes 1 \in \Gamma(U_1, \mathcal{L})$. Let us see what happens outside U_1. Given an arbitrary point q_1 in $X - U_1$ and U_2 an affine open set on the form $D_+(g)$ such that $\mathcal{L}|_{U_2} \approxeq \mathcal{O}_X|_{U_2}$ with $q_1 \in U_2$ ($U_2 \not\subseteq X - U_1$). Let $f_2 \in \Gamma(U_2, \mathcal{L})$ such that corresponds to the one through the isomorphism $\mathcal{L}|_{U_2} \approxeq \mathcal{O}_X|_{U_2}$. On the other hand, by Proposition 2.4 we extend the section $\frac{f_2}{f_1} \in \Gamma(U_1 \cap U_2, \mathcal{K}_X^*)$ to a section $a_2 \in \Gamma(U_2, \mathcal{K}_X^*)$.

Using the same argument and by quasicompacity, we find an affine open finite covering $\{U_2, \ldots, U_q\}$ of $X - U_1$ and sections $\{a_2, \ldots, a_q\}$ of $\mathcal{K}_X^*$ in any of these open sets such that $a_i a_j^{-1} \in \Gamma(U_i \cap U_j, \mathcal{O}_X)$ is a unit. Then the sistem $\{(U_1, f_1), (U_2, a_2), \ldots, (U_q, a_q)\}$ defines a Cartier divisor on X such that $\mathcal{O}_X(D) \approxeq \mathcal{L}$.

3. An invertible sheaf that doesn't come from a Cartier divisor

Now we're going onto our contaurexample. We construct a scheme of dimension three with two closed embedded points that has an invertible sheaf which is not associated to any Cartier divisor. The contaurexemple given by Harsthorne in [Ha] is not correct because in it there's only one embedded point and, in that case, it can be proved that the Cartier divisor class group and the Picard group are isomorphic.

3.1. Let Z be a complete variety over an algebraically closed field that has a 1-cycle $L+M$ numerically equivalent to zero, that is, $L+M=0$. Therefore Z is not a projective variety because the degree funtion in a projective variety is compatible with the numerical equivalence. The existence of such a variety Z has been proved by Nagata and it's shown in [S].

3.2. We define a new sheme Z' that has the same topological space as Z and the same strucural sheaf except in two closed points $p \in L$, $q \in M$ where

$$\mathcal{O}_{Z',p} = \mathcal{O}_{Z,p} \ltimes k(p),\ \mathcal{O}_{Z',q} = \mathcal{O}_{Z,q} \ltimes k(q).$$

with $\ltimes$ denoting the semidirect product.

It turns out that Z is a closed subscheme of Z' since $Z = v(\mathfrak{J})$ where $\mathfrak{J}$ is the nilradical of $\mathcal{O}_{Z'}$. Anyway, $\mathfrak{J}^2 = 0$. If $z : Z \longrightarrow Z'$ is the closed canonical embedding there is a short exact sequence

$$0 \longrightarrow \mathfrak{J} \longrightarrow \mathcal{O}^*_{Z'} \longrightarrow z_*\mathcal{O}^*_Z \longrightarrow 0$$

that induces an long exact sequence of cohomology that gives the isomorphism $\mathrm{Pic}(Z') \overset{z^*}{\approxeq} \mathrm{Pic}(Z)$ making the following diagram commute:

$$\begin{array}{ccccc}
\mathrm{Div}_{\mathrm{C}}(Z') & \longrightarrow & \mathrm{ClCa}(Z') & \longrightarrow & \mathrm{Pic}(Z') \\
{\scriptstyle z^{-1}}\big\downarrow & & \big\downarrow & & {\scriptstyle z^*}\big\downarrow{\scriptstyle \approxeq} \\
\mathrm{Div}_{\mathrm{C}}(Z) & \longrightarrow & \mathrm{ClCa}(Z) & \xrightarrow{\approxeq} & \mathrm{Pic}(Z)
\end{array}$$

3.3. The image of z^{-1} is the set

$$\{D \in \mathrm{Div}_{\mathrm{C}}(Z) \,/\, p, q \notin \mathrm{Supp}(D)\}.$$

Let H be an effective Cartier divisor on Z that intersects M properly, ie, $(H \cdot M_1)_Z > 0$. Let $\mathcal{L} = \mathcal{O}_Z(H)$ and $\mathcal{L}' \in \mathrm{Pic}(Z')$ with $z^*\mathcal{L}' = \mathcal{O}_Z(H)$. We suposse that there exists a divisor $H' \in \mathrm{Div}_{\mathrm{C}}(Z')$ verifying $\mathcal{L}' = \mathcal{O}_{Z'}(H')$ and get a contradition. If we call $H'' = z^{-1}(H')$ we have that $\mathcal{O}_Z(H) \approxeq \mathcal{O}_Z(H'')$, that is, $H \sim_Z H''$. Since the intersection number is invariant under linear equivalence, $(H'' \cdot M)_Z = (H \cdot M)_Z > 0$. And since $p \notin \mathrm{Supp}(H'')$, either H'' intersects L properly or doesn't intersect L. In any case, $(H'' \cdot L)_Z \geq 0$. Therefore $(H'' \cdot (M + L))_Z > 0$, which is a contradiction with the fact that $M + L = 0$.

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