INFINITESIMAL LIFTING AND JACOBI CRITERION FOR SMOOTHNESS ON FORMAL SCHEMES

LEOVIGILDO ALONSO TARRÍO, ANA JEREMÍAS LÓPEZ, AND MARTA PÉREZ RODRÍGUEZ

Abstract. This a first step to develop a theory of smooth, étale and unramified morphisms between noetherian formal schemes. Our main tool is the complete module of differentials, that is a coherent sheaf whenever the map of formal schemes is of pseudo finite type. Among our results we show that these infinitesimal properties of a map of usual schemes carry over into the completion with respect to suitable closed subsets. We characterize unramifiedness by the vanishing of the module of differentials. Also we see that a smooth morphism of noetherian formal schemes is flat and its module of differentials is locally free. The paper closes with a version of Zariski's Jacobian criterion.

Introduction

One of the great achievements of Grothendieck's point of view in algebraic geometry was the relationship between the classical notion of simple point and the notion of infinitesimal lifting. He proved that a point that is "geometrically simple", *i.e.* such that keeps being simple after an extension of base field, can be characterized by the existence of a lifting from any subscheme defined by a square zero ideal of an affine scheme to the full scheme. In recent times formal schemes are getting increasing importance due to the variety of applications in which they are involved, to name a few, as algebraic models of rigid spaces [Raynaud 74], in the study of cohomology of singular spaces [Hartshorne 75] or, more recently, in the context of stable homotopy [Strickland 99]. One feels the need of a greater progress of the basic fundamentals of the theory of formal schemes, so far reduced more or less to the last chapter of [EGA I] and parts of [EGA III_1]. This paper intends to be the first in a series in which infinitesimal conditions on locally noetherian formal schemes are explored together with their applications to cohomology. In a subsequent paper we will give the local structure of smooth and étale maps of formal schemes. In this first installment, we develop a theory of smooth morphisms for noetherian formal schemes. *Chemin faisant*, we also treat the other properties related to infinitesimal lifting, namely étale and unramified morphisms.

2000 *Mathematics Subject Classification.* Primary 14B10; Secondary 14B20, 14B25.

This work was partially supported by Spain's MCyT and E.U.'s FEDER research project MTM2005-05754.

These topics have already been treated in the literature, albeit very scarcely. Smoothness is studied by Yekutieli under a special hypothesis, specifically, condition (ii) in [Yekutieli 98, Definition 2.1] corresponds to a smooth map in which the base is an ordinary noetherian scheme, so smooth formal embeddings are examples of smooth maps of formal schemes. There was also Nayak's 1998 thesis whose results were eventually incorporated to the treatise [Lipman, Nayak, Sastry 2005]. They work in the slightly more general context of essentially pseudo finite type maps (*cf.* [*loc.cit.*, §2.1]). Our work has been developed mostly in parallel to this. As there is some overlapping between this and [*loc.cit.*], we will point it out in the appropriate place when it arises. In fact, both groups of authors have reached an agreement on terminology and their definition of module of differentials [*loc.cit.*, beggining of §2.6] agrees with ours when both are defined.

Let us discuss the contents of this paper. The paper begins with some preliminaries to ease the task of the reader. They are collected into the first paragraph. In the second, we establish the notions that we will study. Our definition is taken from the one in [EGA IV_4, §17.3] for topological algebras. Therefore we will define formal smoothness for a map of formal schemes as the existence of liftings from a map of *ordinary* schemes $T \hookrightarrow Z$ given by a square zero Ideal. This agrees with the definition of formal smoothness for topological algebras and looks very much like the only reasonable convention. We therefore consider the maps like $T \hookrightarrow Z$ as test morphisms for the condition of being formal smooth, unramified or étale. We obtain that maps of formal schemes $\mathfrak{T} \hookrightarrow \mathfrak{Z}$ given by a square zero Ideal also detect formal smoothness (Proposition 2.3). Next we add the condition of being of pseudo finite type to define the notions of smoooth, unramified and étale morphism. Our task is to show that these notions behave in a pleasant way, as in the case of usual schemes. The section closes with the general properties of these notions.

The next section is devoted to the study of the right notion of cotangent bundle for formal schemes. This is the sheaf of differentials that is obtained completing the usual module of differentials. It is our basic tool for studying more advanced properties of smoothness. The definition guarantees that the sheaf of differentials is coherent for a pseudo finite type map of formal schemes. Its basic characterizing property is that together with the canonical derivation it represents the functor that associates to a sheaf of *complete* $\mathcal{O}_{\mathfrak{X}}$-modules, the module of *continuous* derivations. After explaining the functoriality of our construction, we show the analogous of the two fundamental exact sequences in this context.

Once one is equipped with the tool of the module of differentials, one is able to show further properties like the fact that smoothness, unramifiedness and being étale are properties local on the base and also on the source (Proposition 4.1). We show that this properties pass from a map of usual schemes to a completion. We characterize unramifiedness by the vanishing of the module of differentials (Proposition 4.6). Also we see that a smooth morphism is flat and its module of differentials is locally free. Next we discuss the splitting of the fundamental exact sequence when one of the maps is smooth and the chapter closes with Zariski's Jacobian criterion in this context (Corollary 4.15).

Acknowledgments. We have benefited form conversations on these topics and on terminology with Joe Lipman, Suresh Nayak and Pramath Sastry. We also thank José Antonio Álvarez for his useful remarks and the Mathematics Department of Purdue University for hospitality and support.

The diagrams were typeset with Paul Taylor's `diagrams.tex`.

1. Preliminaries

We denote by NFS the category of locally noetherian formal schemes, by $\mathsf{NFS}_{\mathsf{af}}$ the subcategory of affine noetherian formal schemes and by Sch the category of schemes.

We will begin by recalling briefly some basic definitions and results about locally noetherian formal schemes. Of course, for a complete treatment we refer the reader to [EGA I, §10]. We will give some detailed examples of formal schemes, which we will refer along this exposition, like the affine formal scheme and the formal disc. Next we deal with finiteness conditions for morphisms in NFS, which generalize the analogous properties in Sch. In the class of adic morphisms we recall the notions of finite type morphisms, already defined in [EGA I, §10.13]. In the wider class of non adic morphisms we will study morphisms of pseudo finite type (introduced in [Alonso, Jeremías, Lipman 1999, p. 7][1]). Last we will recall from [EGA IV_4, Chapter **0**] some basic properties of the completed module of differentials $\widehat{\Omega}^1_{A/B}$ associated to a continuous morphism $A \to B$ of adic rings.

1.1. [EGA I, (10.2.2) and (10.4.6)] The functors

$$A \rightsquigarrow \operatorname{Spf}(A) \qquad \text{and} \qquad \mathfrak{X} \rightsquigarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$$

define a duality between the category of adic noetherian rings and $\mathsf{NFS}_{\mathsf{af}}$ that generalizes the well-known relation between the categories of rings and affine schemes.

1.2. Every locally noetherian formal scheme is a direct limit of usual schemes and every morphism in NFS is a direct limit of morphisms of schemes. More precisely:

(1) [EGA I, (10.6.3), (10.6.4)] Given $\mathfrak{X}$ in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition, for all $n \in \mathbb{N}$, X_n will denote the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$. Then $\mathfrak{X}$ is the direct limit in NFS of the diagram of noetherian schemes $\{X_n, i_{mn} \colon X_m \hookrightarrow X_n, m \leq n\}_{n \in \mathbb{N}}$. We will recall this data

[1] Morphisms of pseudo finite type have been also introduced independently by Yekutieli in [Yekutieli 98] under the name "formally finite type morphisms"

saying that $\mathfrak{X}$ it is expressed as

$$\mathfrak{X} = \varinjlim_{n\in\mathbb{N}} X_n$$

with respect to the Ideal of definition $\mathcal{J}$ and leave implicit that the schemes $\{X_n\}_{n\in\mathbb{N}}$ are defined by the powers of $\mathcal{J}$.

(2) [EGA I, (10.6.7), (10.6.8) and (10.6.9)] If $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism in NFS, given $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, for each $n \in \mathbb{N}$, $f_n\colon X_n := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1}) \to Y_n := (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^{n+1})$ will be the morphism of schemes induced by f. The morphism f is the direct limit of the system $\{f_n\}_{n\in\mathbb{N}}$ associated to the Ideals of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and we will write

$$f = \varinjlim_{n\in\mathbb{N}} f_n$$

Henceforth we will use systematically the above notations.

1.3. [EGA I, §10.12.] A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ in NFS is *adic* (or simply $\mathfrak{X}$ *is a* $\mathfrak{Y}$*-adic* formal scheme) if there exists an Ideal of definition $\mathcal{K}$ of $\mathfrak{Y}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$ is an Ideal of definition of $\mathfrak{X}$. Note that if there exists an Ideal of definition $\mathcal{K}$ of $\mathfrak{Y}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$ is an Ideal of definition of $\mathfrak{X}$, then all Ideals of definition of $\mathfrak{Y}$ share this property.

If f is adic and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ is an Ideal of definition, then the diagrams of schemes associated to the Ideals $\mathcal{K}$ and $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \uparrow & & \uparrow \\ X_n & \xrightarrow{f_n} & Y_n \end{array} \qquad (m \geq n \geq 0)$$

are cartesian.

Composition of adic morphisms is an adic morphism and the adic property is stable under base-change in NFS.

1.4. [EGA I, §10.14.] Let $\mathfrak{X}$ be in NFS. Given $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$, a coherent Ideal, $\mathfrak{X}' := \operatorname{Supp}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I})$ is a closed subset and $(\mathfrak{X}', (\mathcal{O}_{\mathfrak{X}}/\mathcal{I})|_{\mathfrak{X}'})$ is a locally noetherian formal scheme . We will say that $\mathfrak{X}'$ is the *closed (formal) subscheme* of $\mathfrak{X}$ defined by $\mathcal{I}$.

[EGA I, (10.4.4)] Given $\mathfrak{U} \subset \mathfrak{X}$ open, it holds that $(\mathfrak{U}, \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{U}})$ is a noetherian formal scheme and we say that $\mathfrak{U}$ is an *open subscheme of* $\mathfrak{X}$.

A morphism $f : \mathfrak{Z} \to \mathfrak{X}$ is a *closed immersion* (*open immersion*) if there exists $\mathfrak{Y} \subset \mathfrak{X}$ closed (open, respectively) such that f factors as

$$\mathfrak{Z} \xrightarrow{g} \mathfrak{Y} \hookrightarrow \mathfrak{X}$$

where g is an isomorphism.

Closed and open inmersions are adic morphisms.

Definition 1.5. A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS is of *pseudo finite type* if there exist $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and such that the induced morphism of schemes, $f_0 : X_0 \to Y_0$ is of finite type. If f is of pseudo finite type and adic we say that f is of *finite type* in agreement with [EGA I, (10.13.1)].

We have the following examples of morphisms in $\mathsf{NFS}_{\mathsf{af}}$ provided by 1.1:

Example 1.6. Let A be a J-adic noetherian ring and $\mathbf{T} = T_1, T_2, \dots, T_r$ a finite number of indeterminates.

(1) The ring of restricted formal series $A\{\mathbf{T}\}$ is a $J \cdot A\{\mathbf{T}\}$-adic noetherian ring (*cf.* [EGA I, (**0**, 7.5.2)]). We call $\mathrm{Spf}(A\{\mathbf{T}\})$ the *affine formal r-space over A* or the *affine formal space of dimension r over A* and we will denote it by $\mathbb{A}^r_{\mathrm{Spf}(A)}$. It is a model of the *closed* disk in rigid geometry, *cfr.* [Henrio 2000, §2.2]. Note that $\mathrm{Spf}(A\{\mathbf{T}\}) = \mathrm{Spf}(A) \times \mathrm{Spec}(\mathbb{Z}[\mathbf{T}])$ is the base change on formal schemes of the affine space $\mathrm{Spec}(\mathbb{Z}[\mathbf{T}])$ over $\mathrm{Spec}(\mathbb{Z})$, that is why we adopt this terminology. The canonical projection
$$\mathbb{A}^r_{\mathrm{Spf}(A)} \to \mathrm{Spf}(A)$$
is of finite type.

(2) The formal power series ring $A[[\mathbf{T}]]$ is a $(J \cdot A[[\mathbf{T}]] + \langle \mathbf{T} \rangle \cdot A[[\mathbf{T}]])$-adic noetherian ring (*cf.* [Matsumura 86, Theorem 3.3 and Exercise 8.6]). We define the *formal r-disc over A* or *formal disc of dimension r over A* as $\mathbb{D}^r_{\mathrm{Spf}(A)} = \mathrm{Spf}(A[[\mathbf{T}]])$. It is a model of the *open* disk in rigid geometry, *cfr.* [Henrio 2000, §2.3]. It has no counterpart on usual schemes, so a name relating it to rigid geometry is convenient. The natural projection
$$\mathbb{D}^r_{\mathrm{Spf}(A)} \to \mathrm{Spf}(A)$$
is of pseudo finite type.

(3) Given an ideal $I \subset A$, the closed immersion
$$\mathrm{Spf}(A/I) \hookrightarrow \mathrm{Spf}(A)$$
is a finite type morphism.

(4) Let $a \in A$ and denote by $A_{\{a\}}$ the completion of A_a with respect to the ideal $J \cdot A_a$. The morphism $A \to A_{\{a\}}$ induces the canonical inclusion in $\mathsf{NFS}_{\mathsf{af}}$
$$\mathfrak{D}(a) \hookrightarrow \mathrm{Spf}(A).$$
It is a finite type morphism.

(5) Given $X' = \mathrm{Spec}(A/I)$ a closed subscheme of $X = \mathrm{Spec}(A)$, let $\widehat{A}$ be the completion of A with respect to the I-adic topology. The *morphism of completion of X along X'*, $\kappa : X_{/X'} = \mathrm{Spf}(\widehat{A}) \to X$ is of pseudo finite type and is of finite type only if X and X' have the same underlying topological space hence, $X_{/X'} = X$ and $\kappa = 1_X$.

Proposition 1.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be in* NFS. *The morphism f is of pseudo finite type if, and only if, for each $x \in \mathfrak{X}$, there exist affine open subsets $\mathfrak{V} \subset \mathfrak{Y}$ and $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ and $f(\mathfrak{U}) \subset \mathfrak{V}$ such that $f|_{\mathfrak{U}}$ factors as*

$$\mathfrak{U} \xrightarrow{j} \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{V}}} \xrightarrow{p} \mathfrak{V}$$

where r, $s \in \mathbb{N}$, j is a closed immersion and p is the canonical projection.

If f is of finite type, then the above factorization may be written, taking $s = 0$, $\mathfrak{U} \xrightarrow{j} \mathbb{A}^r_{\mathfrak{V}} \xrightarrow{p} \mathfrak{V}$.

Proof. Since this is a local property we may assume $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$. Given $J \subset A$ and $K \subset B$ ideals of definition such that $KA \subset J$ let $f_0 : X_0 = \mathrm{Spec}(A/J) \to Y_0 = \mathrm{Spec}(B/K)$ be the morphism induced by f. As f is pseudo finite type, there exists a presentation

$$\frac{B}{K} \hookrightarrow \frac{B}{K}[T_1, T_2, \dots, T_r] \overset{\varphi_0}{\twoheadrightarrow} \frac{A}{J}.$$

This morphism lifts to a ring homomorphism

$$B \hookrightarrow B[T_1, T_2, \dots, T_r] \to A$$

that extends to a continuous morphism

$$B \hookrightarrow B\{\mathbf{T}\}[[\mathbf{Z}]] := B\{T_1, T_2, \dots, T_r\}[[Z_1, Z_2, \dots, Z_s]] \xrightarrow{\varphi} A \qquad (1.7.1)$$

such that the images of Z_i in A together with KA generate J. Let $B' := B\{\mathbf{T}\}[[\mathbf{Z}]]$. It is easily seen that the morphism of graded modules associated to φ

$$\bigoplus_{n \in \mathbb{N}} \frac{(KB' + \langle \mathbf{Z} \rangle)^n}{(KB' + \langle \mathbf{Z} \rangle)^{n+1}} \xrightarrow{\mathrm{gr}(\varphi)} \bigoplus_{n \in \mathbb{N}} \frac{J^n}{J^{n+1}}$$

is surjective, therefore, φ is also surjective ([Bourbaki 1989, III, §2.8, Corollary 2]).

If f is of finite type, we may take $K \subset B$ and $J \subset A$ ideals of definition such that $KA = J$, so we can choose $s = 0$. Then, the factorization (1.7.1) may be written

$$B \to B\{T_1, T_2, \dots, T_r\} \twoheadrightarrow A$$

and corresponds with the one given in [EGA I, (10.13.1)]. □

The next result is a general version of [EGA I, (10.3.5)] and follows from the corresponding property in Sch, [EGA I, (6.3.4)].

Proposition 1.8. *We have the following:*

1. *Given $f \colon \mathfrak{X} \to \mathfrak{Y}$ and $g \colon \mathfrak{Y} \to \mathfrak{S}$ in* NFS, *if f and g are (pseudo) finite type morphisms, then $g \circ f$ is a (pseudo) finite type morphism.*
2. *If $f \colon \mathfrak{X} \to \mathfrak{Y}$ is a (pseudo) finite type morphism, given $h \colon \mathfrak{Y}' \to \mathfrak{Y}$ a morphism in* NFS *we have that $\mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ is in* NFS *and that $f' \colon \mathfrak{X}_{\mathfrak{Y}'} \to \mathfrak{Y}'$ is of (pseudo) finite type.*

(3) *Take* $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \to \mathfrak{S}$ *and* $\mathfrak{Y} \xrightarrow{g} \mathfrak{S} \to \mathfrak{S}$ *in* NFS*, such that* $\mathfrak{Y} \times_{\mathfrak{S}} \mathfrak{Y}'$ *is in* NFS*. If* f *and* g *are (pseudo) finite type morphisms, then* $f \times_{\mathfrak{S}} g \colon \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}' \to \mathfrak{Y} \times_{\mathfrak{S}} \mathfrak{Y}'$ *is of (pseudo) finite type.*

Proof. By (1.3) it suffices to prove the assertions for pseudo finite type morphisms. First, (1) and (2) are deduced from the corresponding *sorites* in Sch. Statement (2) follows from the formal argument in [EGA I, (**0**, 1.3.0)]. From this it follows that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}'$ belongs to NFS as a consequence of (1) and (2). Now the result is a consequence of the analogous property in Sch. □

The usual module of differentials of a homomorphism $\phi : A \to B$ of topological algebras is not necessarily complete, but its completion has the good properties of the module of differentials in the discrete case.

1.9. (*cf.* [EGA IV$_1$, §**0**, 20.4, p. 219]) Given $B \to A$ a continuous homomorphism of preadic rings[2] and $K \subset B$, $J \subset A$ ideals of definition such that $KA \subset J$, we denote by $\widehat{\Omega}^1_{A/B}$, the completion of the A-module $\Omega^1_{A/B}$ with respect to the J-adic topology

$$\widehat{\Omega}^1_{A/B} = \varprojlim_{n \in \mathbb{N}} \frac{\Omega^1_{A/B}}{J^{n+1}\Omega^1_{A/B}}.$$

The continuous B-derivation $d_{A/B} : A \to \Omega^1_{A/B}$ extends naturally, by Leibnitz' rule, to a continuous $\widehat{B}$-derivation which, with an abuse of terminology, we will call *canonical complete derivation of* $\widehat{A}$ *over* $\widehat{B}$, and denote by

$$\widehat{d}_{A/B} \colon \widehat{A} \to \widehat{\Omega}^1_{A/B}.$$

The canonical complete derivation of $\widehat{A}$ over $\widehat{B}$ makes the diagram

$$\begin{array}{ccc} A & \xrightarrow{d_{A/B}} & \Omega^1_{A/B} \\ {\scriptstyle can}\big\downarrow & & \big\downarrow{\scriptstyle can} \\ \widehat{A} & \xrightarrow{\widehat{d}_{A/B}} & \widehat{\Omega}^1_{A/B} \end{array}$$

commutative.

For each $n \in \mathbb{N}$ let $A_n = A/J^{n+1}$ and B_n/K^{n+1}. There is a canonical identification

$$\widehat{\Omega}^1_{A/B} \cong \varprojlim_{n \in \mathbb{N}} \Omega^1_{A_n/B_n}$$

with which

$$\widehat{d}_{A/B} \cong \varprojlim_{n \in \mathbb{N}} d_{A_n/B_n}.$$

[2]According to [EGA I, (**0**, 7.1.9)] a ring A is *preadic* if there exists an ideal of definition J of A such that the the collection $\{J^n\}_{n \in \mathbb{N}}$ forms a fundamental system of neighborhoods of 0 in A. If A is moreover separated and complete then A is *adic*.

Remark. Given $B \to A$ a morphism of preadic rings, let $K \subset B$, $J \subset A$ ideals of definition such that $KA \subset J$. As a consequence of the previous discussion there results that

$$(\widehat{\Omega}^1_{A/B}, \widehat{d}_{A/B}) \cong (\widehat{\Omega}^1_{\widehat{A}/\widehat{B}}, \widehat{d}_{\widehat{A}/\widehat{B}}) \tag{1.9.1}$$

where $\widehat{A}$ and $\widehat{B}$ denote the completions of A and B, with respect to the J and K-preadic topologies, respectively, and $\widehat{\Omega}^1_{A/B}$ and $\widehat{\Omega}^1_{\widehat{A}/\widehat{B}}$ denote the completions of $\Omega^1_{A/B}$ and $\Omega^1_{\widehat{A}/\widehat{B}}$, with respect to the J-preadic and $J\widehat{A}$-adic topologies, respectively.

1.10. Let $A\text{-}\mathsf{comp}$ be the category of complete A-modules for the J-adic topology. For all $M \in A\text{-}\mathsf{comp}$ the isomorphism

$$\mathrm{Homcont}_A(\Omega^1_{A/B}, M) \cong \mathrm{Dercont}_B(A, M) \quad (\textit{cf.}\ [\text{EGA IV}_1,\ (\mathbf{0},\ 20.4.8.2)])$$

induces the following canonical isomorphism of B-modules

$$\begin{array}{ccc} \mathrm{Homcont}_A(\widehat{\Omega}^1_{A/B}, M) & \cong & \mathrm{Dercont}_B(\widehat{A}, M) \\ u & \rightsquigarrow & u \circ \widehat{d}_{A/B}. \end{array} \tag{1.10.1}$$

In other words, the pair $(\widehat{\Omega}^1_{A/B}, \widehat{d}_{A/B})$ represents the functor

$$M \in A\text{-}\mathsf{comp} \rightsquigarrow \mathrm{Dercont}_B(\widehat{A}, M).$$

In particular, if M is an A/J-module we have the isomorphism

$$\mathrm{Hom}_A(\widehat{\Omega}^1_{A/B}, M) \cong \mathrm{Der}_B(\widehat{A}, M).$$

1.11. [EGA I, (10.10.1)] Let $\mathfrak{X} = \mathrm{Spf}(A)$ with A a J-adic noetherian ring, $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A/J)$, so we have that $\mathfrak{X} = X_{/X'}$. Given M an A-module, $M^{\triangle}$ denotes the topological $\mathcal{O}_{\mathfrak{X}}$-Module

$$M^{\triangle} := (\widetilde{M})_{/X'} = \varprojlim_{n \in \mathbb{N}} \frac{\widetilde{M}}{\widetilde{J}^{n+1}\widetilde{M}}.$$

Moreover, a morphism $u : M \to N$ in $A\text{-}\mathsf{mod}$ corresponds to a morphism of $\mathcal{O}_X$-Modules $\tilde{u} : \widetilde{M} \to \widetilde{N}$ that induces a morphism of $\mathcal{O}_{\mathfrak{X}}$-Modules

$$M^{\triangle} \xrightarrow{u^{\triangle}} N^{\triangle} \quad = \quad \varprojlim_{n \in \mathbb{N}} \Big(\frac{\widetilde{M}}{\widetilde{J}^{n+1}\widetilde{M}} \xrightarrow{\tilde{u}_n} \frac{\widetilde{N}}{\widetilde{J}^{n+1}\widetilde{N}}\Big).$$

So there is an additive covariant functor from the category of A-modules to the category of $\mathcal{O}_{\mathfrak{X}}$-Modules

$$\begin{array}{ccc} A\text{-}\mathsf{mod} & \xrightarrow{\triangle} & \mathsf{Mod}(\mathfrak{X}) \\ M & \rightsquigarrow & M^{\triangle}. \end{array} \tag{1.11.1}$$

1.12. If $M \in A\text{-}\mathsf{mod}$ and $\widehat{M}$ denotes the complete module of M for the J-adic topology, from the definition of the functor $\triangle$ it is easy to deduce that:

(1) (*Cf.* [EGA I, proof of (10.10.2.1)]) $\Gamma(\mathfrak{X}, M^{\triangle}) = \widehat{M}$.
(2) For all $a \in A$, $\Gamma(\mathfrak{D}(a), M^{\triangle}) = M_{\{a\}}$.
(3) [EGA I, (10.10.2)] The functor $(-)^{\triangle}$ defines an equivalence of categories between finite type A-modules and the category $\mathsf{Coh}(\mathfrak{X})$ of coherent $\mathcal{O}_{\mathfrak{X}}$-Modules.
(4) [EGA I, (10.10.2.1)] The functor $(-)^{\triangle}$ is exact on the category of A-modules of finite type.

1.13. A consequence of the previous results is the following. Let us consider a morphism $f\colon \mathrm{Spf}(A) \to \mathrm{Spf}(B)$ in $\mathsf{NFS}_{\mathsf{af}}$. Let $a \in A$ and $b \in B$ such that $f(\mathrm{Spf}(A)_{\{a\}}) \subset \mathrm{Spf}(B)_{\{b\}}$, then

$$(\widehat{\Omega}^1_{A/B})^{\triangle}(\mathrm{Spf}(A)_{\{a\}}) = (\Omega^1_{A/B})_{\{a\}} = \widehat{\Omega}^1_{A_a/B} = \widehat{\Omega}^1_{A_a/B_b} = \widehat{\Omega}^1_{A_{\{a\}}/B_{\{b\}}}.$$

Therefore, the sheaf $(\widehat{\Omega}^1_{A/B})^{\triangle}$ agrees with the presheaf defined on principal open subsets by $\mathrm{Spf}(A)_{\{a\}} \rightsquigarrow \widehat{\Omega}^1_{A_{\{a\}}/B}$.

2. Definitions of the infinitesimal lifting properties

In this section we extend Grothendieck's classical definition of infinitesimal lifting properties from the category of schemes (*cf.* [EGA IV_4, (17.1.1)]) to the category of locally noetherian formal schemes and we present some of their basic properties. We will refer to a morphism of formal schemes $f\colon \mathfrak{X} \to \mathfrak{Y}$ simply as a $\mathfrak{Y}$-formal scheme if there is no risk of ambiguity. If $X = \mathfrak{X}$ is an ordinary scheme, we will say that X is a $\mathfrak{Y}$-scheme.

Definition 2.1. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in NFS. We say that f is *formally smooth (formally unramified or formally étale)* if it satisfies the following lifting condition:

For all affine $\mathfrak{Y}$-scheme Z and for each closed subscheme $T \hookrightarrow Z$ given by a square zero Ideal $\mathcal{I} \subset \mathcal{O}_Z$ the induced map

$$\mathrm{Hom}_{\mathfrak{Y}}(Z, \mathfrak{X}) \longrightarrow \mathrm{Hom}_{\mathfrak{Y}}(T, \mathfrak{X}) \tag{2.1.1}$$

is surjective (injective or bijective, respectively).

So, f is formally étale if, and only if, is formally smooth and formally unramified.

2.2. Let $f : \mathrm{Spf}(A) \to \mathrm{Spf}(B)$ be in $\mathsf{NFS}_{\mathsf{af}}$. Applying (1.1), we obtain that f is formally smooth (formally unramified or formally étale) if, and only if, the topological B-algebra A is formally smooth (formally unramified or formally étale, respectively) (*cf.* [EGA IV_1, (**0**, 19.3.1) and (**0**, 19.10.2)]).

The reference for basic properties of the infinitesimal lifting conditions on preadic rings is [EGA IV_1, **0**, §§ 19.3 and 19.10].

Next proposition shows that the lifting condition (2.1.1) extends to a wider class of test maps.

Proposition 2.3. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be in* NFS. *If f is formally smooth (formally unramified or formally étale), then for all affine noetherian $\mathfrak{Y}$-formal scheme $\mathfrak{Z}$ and for all closed formal subschemes $\mathfrak{T} \hookrightarrow \mathfrak{Z}$ given by a square zero Ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Z}}$, the induced map*

$$\operatorname{Hom}_{\mathfrak{Y}}(\mathfrak{Z}, \mathfrak{X}) \longrightarrow \operatorname{Hom}_{\mathfrak{Y}}(\mathfrak{T}, \mathfrak{X}) \tag{2.3.1}$$

is surjective (injective or bijective, respectively).

Proof. Let $\mathfrak{T} = \operatorname{Spf}(C/I) \overset{j}{\hookrightarrow} \mathfrak{Z} = \operatorname{Spf}(C)$ be a closed formal subscheme given by a square zero ideal $I \subset C$. Let $L \subset C$ be an ideal of definition, writing $T_n = \operatorname{Spec}(C/(I + L^{n+1}))$ and $Z_n = \operatorname{Spec}(C/L^{n+1})$, the embedding $j \colon \mathfrak{T} \hookrightarrow \mathfrak{Z}$ is expressed as (see 1.2.(2))

$$\varinjlim_{n \in \mathbb{N}} (T_n \overset{j_n}{\hookrightarrow} Z_n),$$

where the morphisms j_n are closed immersions of affine schemes defined by a square zero Ideal. Given $u : \mathfrak{T} \to \mathfrak{X}$ a $\mathfrak{Y}$-morphism, we will denote by u'_n the morphisms $T_n \hookrightarrow \mathfrak{T} \xrightarrow{u} \mathfrak{X}$ that make the diagrams

$$\begin{array}{ccc} T_n & \overset{j_n}{\hookrightarrow} & Z_n \\ {\scriptstyle u'_n}\downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

commutative for all $n \in \mathbb{N}$.

Suppose that f is formally smooth. Translating the argument given in [EGA IV_1, (**0**, 19.3.10)] for topological algebras to the context of formal schemes we get a $\mathfrak{Y}$-morphism

$$v := \varinjlim_{n \in \mathbb{N}} (v'_n : Z_n \to \mathfrak{X})$$

that satisfies $v|_{\mathfrak{T}} = u$. The morphisms $\{v'_n\}_{n \in \mathbb{N}}$ are constructed by induction and satisfy that $v'_n|_{T_n} = u'_n$ and $v'_n|_{Z_{n-1}} = v'_{n-1}$, for each $n > 0$ (*cf.* [EGA IV_1, (**0**, 19.3.10.1) and (**0**, 19.3.10.2)]).

If f is formally unramified, assume there exist $\mathfrak{Y}$-morphisms $v : \mathfrak{Z} \to \mathfrak{X}$ and $w : \mathfrak{Z} \to \mathfrak{X}$ such that $v|_{\mathfrak{T}} = w|_{\mathfrak{T}} = u$. With the notations established at the beginning of the proof consider

$$v = \varinjlim_{n \in \mathbb{N}} (v'_n : Z_n \to \mathfrak{X}) \quad \text{and} \quad w = \varinjlim_{n \in \mathbb{N}} (w'_n : Z_n \to \mathfrak{X})$$

such that the diagram

$$\begin{array}{ccccc} T_n & \overset{j_n}{\hookrightarrow} & Z_n & & \\ & {\scriptstyle u'_n}\searrow & {\scriptstyle v'_n}\downarrow\downarrow{\scriptstyle w'_n} & \searrow & \\ & & \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

commutes. By hypothesis we have that $v'_n = w'_n$, for all $n \in \mathbb{N}$, and we conclude that

$$v = \varinjlim_{n\in\mathbb{N}} v'_n = \varinjlim_{n\in\mathbb{N}} w'_n = w. \qquad \square$$

2.4. In Definition 2.1 the test morphisms for the lifting condition are closed subschemes of *affine* $\mathfrak{Y}$-schemes given by square-zero ideals. An easy patching argument gives that the uniqueness of lifting conditions holds for closed subschemes of *arbitrary* $\mathfrak{Y}$-schemes given by square-zero ideals ([EGA IV$_4$, (17.1.2.(iv))]). This applies to formally unramified and formally étale morphisms.

Corollary 2.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be in* NFS. *If the morphism f is formally unramified (or formally étale), then for all noetherian $\mathfrak{Y}$-formal schemes $\mathfrak{Z}$ and for each closed formal subscheme $\mathfrak{T} \hookrightarrow \mathfrak{Z}$ given by a square zero Ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Z}}$, the induced map*

$$\operatorname{Hom}_{\mathfrak{Y}}(\mathfrak{Z}, \mathfrak{X}) \longrightarrow \operatorname{Hom}_{\mathfrak{Y}}(\mathfrak{T}, \mathfrak{X}) \tag{2.5.1}$$

is injective (or bijective, respectively).

Proof. Given $\{\mathfrak{V}_\alpha\}$ a covering of affine open formal subschemes of $\mathfrak{Z}$, denote by $\{\mathfrak{U}_\alpha\}$ the covering of affine open formal subschemes of $\mathfrak{T}$ given by $\mathfrak{U}_\alpha = \mathfrak{V}_\alpha \cap \mathfrak{T}$, for all α. By [EGA I, (10.14.4)] $\mathfrak{U}_\alpha \hookrightarrow \mathfrak{V}_\alpha$ is a closed immersion in NFS determined by a square zero Ideal. Therefore, the proof follows the same line as [EGA IV$_4$, (17.1.2.(iv))]. $\square$

The study of infinitesimal properties in Sch using the module of differentials leads one to look at the class of finite type morphisms. Under this assumption there are nice characterizations of the infinitesimal lifting conditions in terms of the module of differentials. We will consider two conditions for morphisms in NFS that generalize the property of being of finite type for morphisms in Sch: morphisms of pseudo finite type ([Alonso, Jeremías, Lipman 1999, 1.2.2]) and its adic counterpart, morphisms of finite type ([EGA I, (10.13.3)]).

Definition 2.6. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be in NFS. The morphism f is *smooth (unramified or étale)* if, and only if, it is of pseudo finite type and formally smooth (formally unramified or formally étale, respectively). If moreover f is adic, we say that f is *adic smooth (adic unramified or adic étale, respectively).* So f is adic smooth (adic unramified or adic étale) if it is of finite type and formally smooth (formally unramified or formally étale, respectively).

If $f : X \to Y$ is in Sch, both definitions agree with the one given in [EGA IV$_4$, (17.3.1)] and we say that f is smooth (unramified or étale, respectively).

Using 2.2 we will be able to describe a few basic examples of morphisms in $\mathsf{NFS}_{\mathsf{af}}$ that satisfy some of the infinitesimal lifting conditions (Example

2.8). Before all else, let us recall some of the properties of the infinitesimal lifting conditions for preadic rings.

Remark. Let $B \to A$ be a continuous morphism of preadic rings and take $J \subset A$, $K \subset B$ ideals of definition with $KA \subset J$. Given $J' \subset A$, $K' \subset B$ ideals such that $K'A \subset J'$, $J \subset J'$ and $K \subset K'$, if A is a formally smooth (formally unramified or formally étale) B-algebra for the J and K-adic topologies then, we have that A is a formally smooth (formally unramified or formally étale, respectively) B-algebra for the J' and K'-adic topologies, respectively.

Lemma 2.7. *Let $B \to A$ be a continuous morphism of preadic rings, $J \subset A$ and $K \subset B$ ideals of definition with $KA \subset J$ and let us denote by $\widehat{A}$ and $\widehat{B}$ the respective completions of A and B. The following conditions are equivalent:*

(1) *A is a formally smooth (formally unramified or formally étale) B-algebra*
(2) *$\widehat{A}$ is a formally smooth (formally unramified or formally étale, respectively) B-algebra*
(3) *$\widehat{A}$ is a formally smooth (formally unramified or formally étale, respectively) $\widehat{B}$-algebra*

Proof. It suffices to note that

$$\mathrm{Homcont}_{B\text{-}\mathsf{Alg}}(A, C) \cong \mathrm{Homcont}_{B\text{-}\mathsf{Alg}}(\widehat{A}, C) \cong \mathrm{Homcont}_{\widehat{B}\text{-}\mathsf{Alg}}(\widehat{A}, C)$$

for all discrete rings C and all continuous homomorphisms $B \to C$. □

Example 2.8. Put $\mathfrak{X} = \mathrm{Spf}(A)$ with A a J-adic noetherian ring and let $\mathbf{T} = T_1, T_2, \dots, T_r$ be a finite number of indeterminates.

(1) If we take in A the discrete topology, from the universal property of the polynomial ring it follows that $A[\mathbf{T}]$ is a formally smooth A-algebra. Applying the previous remark and Lemma 2.7 we have that the restricted formal series ring $A\{\mathbf{T}\}$ is a formally smooth A-algebra, therefore the canonical morphism $\mathbb{A}^r_{\mathfrak{X}} \to \mathfrak{X}$ is adic smooth.
(2) Analogously to the preceding example, we obtain that $A[[\mathbf{T}]]$ is a formally smooth A-algebra, from which we deduce that projection $\mathbb{D}^r_{\mathfrak{X}} \to \mathfrak{X}$ is smooth.
(3) If we take in A the discrete topology it is known that, given $a \in A$, A_a is a formally étale A-algebra. So, there results that the canonical inclusion $\mathfrak{D}(a) \hookrightarrow \mathfrak{X}$ is adic étale.
(4) Trivially, every surjective morphism of rings is formally unramified. Therefore, given an ideal $I \subset A$, the closed immersion $\mathrm{Spf}(A/I) \hookrightarrow \mathfrak{X}$ is adic unramified.
(5) If $X' = \mathrm{Spec}(A/I)$ is a closed subscheme of X, $\kappa : X_{/X'} \to X$, the morphism of completion of X along X', corresponds through (1.1) with the continuous morphism of rings $A \to \widehat{A}$, where $\widehat{A}$ is the completion of A for the I-adic topology and therefore, κ is étale.

Proposition 2.9. *In the category* NFS *of locally noetherian formal schemes the following properties hold:*

1. *Composition of smooth (unramified or étale) morphisms is a smooth (unramified or étale, respectively) morphism.*
2. *Smooth, unramified and étale character is stable under base-change in* NFS*.*
3. *Product of smooth (unramified or étale) morphisms is a smooth (unramified or étale, respectively) morphism.*

Proof. Keeping in mind that composition of pseudo finite type maps is a pseudo finite type map and pseudo finite type character of a map is preserved under base-change (Proposition 1.8) the proof is similar to [EGA IV_4, (17.1.3) (ii), (iii) and (iv)] □

Proposition 2.10. *The assertions of the last proposition hold if we change the infinitesimal conditions by the corresponding infinitesimal adic conditions.*

Proof. By the definition of the infinitesimal adic conditions, it suffices to apply the last result and the *sorites* of finite type morphisms (Proposition 1.8). □

Example 2.11. Let $\mathfrak{X}$ be in NFS and $r \in \mathbb{N}$. From Proposition 2.9.(2), Example 2.8.(1) and Example 2.8.(2) we get that:

1. The morphism of projection $\mathbb{A}^r_{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{A}^r_{\mathrm{Spec}(\mathbb{Z})} \to \mathfrak{X}$ is an adic smooth morphism.
2. The canonical morphism $\mathbb{D}^r_{\mathfrak{X}} := \mathfrak{X} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{D}^r_{\mathrm{Spec}(\mathbb{Z})} \to \mathfrak{X}$ is smooth.

Proposition 2.12. *The following holds in the category of locally noetherian formal schemes:*

1. *A closed immersion is adic unramified.*
2. *An open immersion is adic étale.*

Proof. Closed and open immersions (see 1.4) are monomorphisms and therefore, unramified. On the other hand, open immersions are smooth morphisms. □

Proposition 2.13. *Let $f : \mathfrak{X} \to \mathfrak{Y}$, $g : \mathfrak{Y} \to \mathfrak{S}$ be two morphisms of pseudo finite type in* NFS*.*

1. *If $g \circ f$ is unramified, then so is f.*
2. *Let us suppose that g is unramified. If $g \circ f$ is smooth (or étale) then, f is smooth (or étale, respectively).*

Proof. Item (1) is immediate. The proof of (2) is analogous to [EGA IV_4, (17.1.4)]. □

Corollary 2.14. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo finite type morphism and $g : \mathfrak{Y} \to \mathfrak{S}$ an étale morphism. The morphism $g \circ f$ is smooth (or étale) if, and only if, f is smooth (or étale, respectively).*

Proof. It is enough to apply Proposition 2.9.(1) and Proposition 2.13. □

3. Differentials of a pseudo finite type map of formal schemes

Given $f : X \to Y$ a finite type morphism of schemes, it is well-known that $\Omega^1_{X/Y}$, the module of 1-differentials of X over Y, is an essential tool to study the smooth, unramified or étale character of f. In this section, we introduce the module of 1-differentials for a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS and discuss its fundamental properties, which will be used in the characterizations of the infinitesimal conditions in Section 4. We cannot use the general definition for ringed spaces, because it does not take into account the topology in the structure sheaves.

The observation in 1.13 shows that the following definition makes sense.

Definition 3.1. Given $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS we call *module of* 1*-differentials of* f or *module of* 1*-differentials of* $\mathfrak{X}$ *over* $\mathfrak{Y}$ and we will denote it by $\widehat{\Omega}^1_f$ or $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$, the sheaf of topological $\mathcal{O}_{\mathfrak{X}}$-Modules locally given by $(\widehat{\Omega}^1_{A/B})^{\triangle}$ (see 1.13), for all open sets $\mathfrak{U} = \operatorname{Spf}(A) \subset \mathfrak{X}$ and $\mathfrak{V} = \operatorname{Spf}(B) \subset \mathfrak{Y}$ with $f(\mathfrak{U}) \subset \mathfrak{V}$. Note that $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ has structure of $\mathcal{O}_{\mathfrak{X}}$-Module.

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. These Ideals provide us with an inverse system of derivations

$$d_{\mathfrak{X}_n/\mathfrak{Y}_n} \colon \frac{\mathcal{O}_{\mathfrak{X}}}{\mathcal{I}^{n+1}} \to \Omega^1_{\mathfrak{X}_n/\mathfrak{Y}_n}, \quad n \in \mathbb{N}.$$

Let $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}} : \mathcal{O}_{\mathfrak{X}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ be the morphism

$$\varprojlim_{n\in\mathbb{N}} d_{X_n/Y_n} = \varprojlim_{n\in\mathbb{N}} \big(\frac{\mathcal{O}_{\mathfrak{X}}}{\mathcal{I}^{n+1}} \xrightarrow{d_{\mathfrak{X}_n/\mathfrak{Y}_n}} \Omega^1_{\mathfrak{X}_n/\mathfrak{Y}_n}\big).$$

It is locally defined for all couple of affine open sets $\mathfrak{U} = \operatorname{Spf}(A) \subset \mathfrak{X}$ and $\mathfrak{V} = \operatorname{Spf}(B) \subset \mathfrak{Y}$ such that $f(\mathfrak{U}) \subset \mathfrak{V}$ by $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}(\operatorname{Spf}(A)) = \widehat{d}_{A/B} \colon A \to \widehat{\Omega}^1_{A/B}$. This construction is independent of the Ideals of definition chosen for $\mathfrak{X}$ and $\mathfrak{Y}$, see 1.9.

The morphism $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$ is a continuous $\mathfrak{Y}$-derivation and it is called the *canonical derivation of* $\mathfrak{X}$ *over* $\mathfrak{Y}$. We will refer to $(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \widehat{d}_{\mathfrak{X}/\mathfrak{Y}})$ as the *differential pair of* $\mathfrak{X}$ *over* $\mathfrak{Y}$.

3.2. If $X = \operatorname{Spec}(A) \to Y = \operatorname{Spec}(B)$ is a morphism of usual schemes, there results that $(\widehat{\Omega}^1_{X/Y}, \widehat{d}_{X/Y}) = (\Omega^1_{X/Y}, d_{X/Y})$ is the differential pair of the morphism of affine schemes (*cf.* [EGA IV$_4$, (16.5.3)]).

Remark. Our definition of the differential pair, $(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \widehat{d}_{\mathfrak{X}/\mathfrak{Y}})$, of a morphism $\mathfrak{X} \to \mathfrak{Y}$ in NFS, agrees with the one given in [Lipman, Nayak, Sastry 2005, 2.6] where it is directly defined as

$$\varprojlim_{n\in\mathbb{N}} \big(\frac{\mathcal{O}_{\mathfrak{X}}}{\mathcal{I}^{n+1}} \xrightarrow{d_{\mathfrak{X}_n/\mathfrak{Y}_n}} \Omega^1_{\mathfrak{X}_n/\mathfrak{Y}_n}\big).$$

Proposition 3.3. *(*cf. [Lipman, Nayak, Sastry 2005, Proposition 2.6.1]*) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in* NFS *of pseudo finite type. Then $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is a coherent sheaf.*

Proof. We may suppose that $f : \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathsf{NFS}_{\mathsf{af}}$. Let $J \subset A$ and $K \subset B$ be ideals of definition such that $KA \subset J$. By hypothesis we have that $B_0 = B/K \to A_0 = A/J$ is a finite type morphism and therefore, $\Omega^1_{A_0/B_0}$ is a finite type A_0-module. From [EGA IV$_1$, (**0**, 20.7.15)]) it follows that $\widehat{\Omega}^1_{A/B}$ is a finite type A-module. Therefore, since $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = (\widehat{\Omega}^1_{A/B})^{\triangle}$ the result is deduced from 1.12.3. □

Given $X \to Y$ a morphism of schemes in [EGA IV$_4$, (16.5.3)] it is established that $(\Omega^1_{X/Y}, d_{X/Y})$ is the universal pair of the representable functor $\mathcal{F} \in \mathsf{Mod}(X) \rightsquigarrow \mathrm{Der}_Y(\mathcal{O}_X, \mathcal{F})$. In Theorem 3.5 this result is generalized for a morphism $\mathfrak{X} \to \mathfrak{Y}$ in NFS.

3.4. Given $\mathfrak{X}$ in NFS and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ an Ideal of definition of $\mathfrak{X}$ we will denote by $\mathsf{Comp}(\mathfrak{X})$ the full subcategory of $\mathcal{O}_{\mathfrak{X}}$-Modules $\mathcal{F}$ such that

$$\mathcal{F} = \varprojlim_{n \in \mathbb{N}} (\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}).$$

It is easily seen that the definition does not depend on the election of the Ideal of definition of $\mathfrak{X}$.

For example:

(1) Given $\mathfrak{X} = \mathrm{Spf}(A)$ in $\mathsf{NFS}_{\mathsf{af}}$ and $J \subset A$ an ideal of definition for all A-modules M, it holds that

$$M^{\triangle} = \varprojlim_{n \in \mathbb{N}} \frac{\widetilde{M}}{\widetilde{J}^{n+1}\widetilde{M}} \in \mathsf{Comp}(\mathfrak{X}).$$

(2) Let $\mathfrak{X}$ be in NFS. For all $\mathcal{F} \in \mathsf{Coh}(\mathfrak{X})$, we have that

$$\mathcal{F} = \varprojlim_{n \in \mathbb{N}} (\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n})$$

by [EGA I, (10.11.3)] and therefore, $\mathsf{Coh}(\mathfrak{X})$ is a full subcategory of $\mathsf{Comp}(\mathfrak{X})$. Consequently, if $f : \mathfrak{X} \to \mathfrak{Y}$ is a pseudo finite type morphism in NFS, then $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \in \mathsf{Comp}(\mathfrak{X})$ by (3.3).

Now we are ready to show that given $\mathfrak{X} \to \mathfrak{Y}$ a morphism in NFS, $(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \widehat{d}_{\mathfrak{X}/\mathfrak{Y}})$ is the universal pair for the representable functor

$$\mathcal{F} \in \mathsf{Comp}(\mathfrak{X}) \rightsquigarrow \mathrm{Dercont}_{\mathfrak{Y}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{F}).$$

Theorem 3.5. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in* NFS*. Then the canonical map*

$$\begin{array}{ccc} \mathrm{Homcont}_{\mathcal{O}_{\mathfrak{X}}}(\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}, \mathcal{F}) & \xrightarrow{\varphi} & \mathrm{Dercont}_{\mathfrak{Y}}(\mathcal{O}_{\mathfrak{X}}, \mathcal{F}) \\ u & \rightsquigarrow & u \circ \widehat{d}_{\mathfrak{X}/\mathfrak{Y}} \end{array}$$

is an isomorphism for every $\mathcal{F} \in \mathsf{Comp}(\mathfrak{X})$.

Proof. It is a globalization of [EGA IV_1, (**0**, 20.7.14.4)]. We leave the details to the reader. □

Lemma 3.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in* NFS*. If $\mathcal{F} \in \mathsf{Comp}(\mathfrak{X})$ then*

$$f_*\mathcal{F} = \varprojlim_{n\in\mathbb{N}} (f_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{Y}} \mathcal{O}_{Y_n})$$

and consequently, $f_*\mathcal{F}$ is in $\mathsf{Comp}(\mathfrak{Y})$.

Proof. Let $\mathcal{J} \subset \mathcal{O}_\mathfrak{X}$ and $\mathcal{K} \subset \mathcal{O}_\mathfrak{Y}$ be Ideals of definition such that $f^*(\mathcal{K})\mathcal{O}_\mathfrak{X} \subset \mathcal{J}$. For all $n \in \mathbb{N}$ we have the canonical morphisms $f_*\mathcal{F} \to f_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{Y}} \mathcal{O}_{Y_n}$ that induce the morphism of $\mathcal{O}_\mathfrak{Y}$-Modules

$$f_*\mathcal{F} \longrightarrow \varprojlim_{n\in\mathbb{N}} (f_*\mathcal{F} \otimes_{\mathcal{O}_\mathfrak{Y}} \mathcal{O}_{Y_n})$$

To see whether it is an isomorphism is a local question, therefore we may assume that $f = \mathrm{Spf}(\phi)\colon \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{Y} = \mathrm{Spf}(B)$ is in $\mathsf{NFS}_{\mathsf{af}}$, $\mathcal{J} = J^\triangle$ and $\mathcal{K} = K^\triangle$ with $J \subset A$ and $K \subset B$ ideals of definition such that $KA \subset J$. Then $M = \Gamma(\mathfrak{Y}, f_*\mathcal{F})$ is a complete B-module for the $\phi^{-1}(J)$-adic topology and since $K \subset \phi^{-1}(J)$ we have that

$$M = \varprojlim_{n\in\mathbb{N}} \frac{M}{K^{n+1}M}$$

and the result follows. □

Proposition 3.7. *Given a commutative diagram in* NFS *of pseudo finite type morphisms*

$$\begin{array}{ccc} \mathfrak{X} & \longrightarrow & \mathfrak{Y} \\ {\scriptstyle g}\uparrow & & \uparrow \\ \mathfrak{X}' & \xrightarrow{h} & \mathfrak{Y}' \end{array}$$

there exists a morphism of $\mathcal{O}_{\mathfrak{X}'}$-Modules $g^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \longrightarrow \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'}$ locally determined by $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}(a) \otimes 1 \rightsquigarrow \widehat{d}_{\mathfrak{X}'/\mathfrak{Y}'}g(a)$. Moreover, if the diagram is cartesian, the above morphism is an isomorphism.

Proof. The morphism

$$\mathcal{O}_\mathfrak{X} \to g_*\mathcal{O}_{\mathfrak{X}'} \xrightarrow{g_*\widehat{d}_{\mathfrak{X}'/\mathfrak{Y}'}} g_*\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'}$$

is a continuous $\mathfrak{Y}$-derivation. Applying Proposition 3.3 and Lemma 3.6 we have that $g_*\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'} \in \mathsf{Comp}(\mathfrak{X})$ and therefore by Theorem 3.5 there exists an unique morphism of $\mathcal{O}_\mathfrak{X}$-Modules $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to g_*\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'}$ such that the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O}_{\mathfrak{X}} & \xrightarrow{\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}} & \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \\ \downarrow & & \downarrow \\ g_*\mathcal{O}_{\mathfrak{X}'} & \xrightarrow{g_*\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}} & g_*\widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'} \end{array}$$

Equivalently, there exists a morphism of $\mathcal{O}_{\mathfrak{X}'}$-Modules $g^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}'}$ locally determined by $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}a \otimes 1 \rightsquigarrow \widehat{d}_{\mathfrak{X}'/\mathfrak{Y}'}g(a)$.

Let us suppose that the square of formal schemes in the statement of this proposition is cartesian. We may assume $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, $\mathfrak{Y}' = \mathrm{Spf}(B')$ and $\mathfrak{X}' = \mathrm{Spf}(A')$ with $A' = A\widehat{\otimes}_B B'$. The induced topology in $\widehat{\Omega}_{A/B} \otimes_A A'$ is the one given by the topology of A'. As a consequence of the canonical isomorphism of A'-modules $\Omega^1_{A'/B'} \cong \Omega^1_{A/B} \otimes_A A'$ (*cf.* [EGA IV$_1$, (**0**, 20.5.5)]) it holds that

$$\widehat{\Omega}^1_{A'/B'} \cong \widehat{\Omega}^1_{A\widehat{\otimes}_B B'/B'} \underset{(1.9)}{\cong} \Omega^1_{A/B}\widehat{\otimes}_A A' \underset{[\text{EGA I}, (\mathbf{0}, 7.7.1)]}{\cong} \widehat{\Omega}^1_{A/B}\widehat{\otimes}_A A'.$$

Finally $\widehat{\Omega}^1_{A/B}$ is an A-module of finite type (see Proposition 3.3) hence, $\widehat{\Omega}^1_{A'/B'} \cong \widehat{\Omega}^1_{A/B} \otimes_A A'$. □

With the previous notations, if $\mathfrak{Y} = \mathfrak{Y}'$ the morphism $g^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \longrightarrow \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}}$ is denoted by dg and is called *the differential of g over $\mathfrak{Y}$*.

Corollary 3.8. *Given $f : \mathfrak{X} \to \mathfrak{Y}$ a* finite type *morphism in* NFS *consider $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ and $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{O}_{\mathfrak{X}}$ Ideals of definition that let us express*

$$f = \varinjlim_{n\in\mathbb{N}} (f_n : X_n \to Y_n).$$

Then

$$\Omega^1_{X_n/Y_n} \cong \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}.$$

for all $n \in \mathbb{N}$.

Proof. Since f is an adic morphism, the diagrams

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ \uparrow & & \uparrow \\ X_n & \xrightarrow{f_n} & Y_n \end{array}$$

are cartesian, $\forall n \geq 0$. Then the corollary follows. □

In the following example we show that if the morphism is not adic, the last corollary does not hold.

Example 3.9. Let K be a field and $p : \mathbb{D}^1_K \to \operatorname{Spec}(K)$ the projection morphism of the formal disc of 1 dimension over $\operatorname{Spec}(K)$. Given the ideal of definition $\langle T\rangle \subset K[[T]]$ such that

$$p = \varinjlim_{n\in\mathbb{N}} p_n$$

we have that $\Omega^1_{p_0} = 0$ but,

$$\widehat{\Omega}^1_p \otimes_{\mathcal{O}_{\mathbb{D}^1_K}} \mathcal{O}_{\operatorname{Spec}(K)} = (\widehat{\Omega}^1_{K[[T]]/K})^{\triangle} \otimes_{K[[T]]^{\triangle}} \widetilde{K} \cong \widetilde{K} \neq 0.$$

We extend the usual First and Second Fundamental Sequences to our construction of differentials of pseudo finite type morphisms between formal schemes. They will provide a basic tool for applying it to the study of the infinitesimal lifting. Also, we will give a local computation based on the Second Fundamental Exact Sequence.

Proposition 3.10. *(First Fundamental Exact Sequence) Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ and $g\colon \mathfrak{Y} \to \mathfrak{S}$ be two morphisms in* NFS *of pseudo finite type. There exists an exact sequence of coherent $\mathcal{O}_{\mathfrak{X}}$-Modules*

$$f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \xrightarrow{\Phi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\Psi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0 \tag{3.10.1}$$

where Φ and Ψ are locally defined by

$$\widehat{d}_{\mathfrak{Y}/\mathfrak{S}} b \otimes 1 \rightsquigarrow \widehat{d}_{\mathfrak{X}/\mathfrak{S}} f(b) \qquad \widehat{d}_{\mathfrak{X}/\mathfrak{S}} a \rightsquigarrow \widehat{d}_{\mathfrak{X}/\mathfrak{Y}} a$$

Proof. This is a globalization of [Lipman, Nayak, Sastry 2005, Lemma 2.5.2] (see also [EGA IV$_1$, (**0**, 20.7.17.3)]). The morphism $\Phi\colon f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ of $\mathcal{O}_{\mathfrak{X}}$-Modules is df, the differential of f over $\mathfrak{S}$. Since $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}\colon \mathcal{O}_{\mathfrak{X}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is a continuous $\mathfrak{S}$-derivation, from Theorem 3.5 there exists a unique morphism of $\mathcal{O}_{\mathfrak{X}}$-Modules $\Psi\colon \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ such that $\Psi \circ \widehat{d}_{\mathfrak{X}/\mathfrak{S}} = \widehat{d}_{\mathfrak{X}/\mathfrak{Y}}$.

As for proving the exactness we can reduce to the affine case and then it is the first part of [Lipman, Nayak, Sastry 2005, Lemma 2.5.5] (see also [EGA IV$_1$, (**0**, 20.7.17)]). □

3.11. Let $f\colon \mathfrak{X} \to \mathfrak{Y}$ be a pseudo finite type morphism in NFS, and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{K} \subset \mathcal{O}_{\mathfrak{Y}}$ be Ideals of definition with $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ and

$$f\colon \mathfrak{X} \to \mathfrak{Y} = \varinjlim_{n\in\mathbb{N}} (f_n\colon X_n \to Y_n)$$

the relevant expression for f. For all $n \in \mathbb{N}$, from the First Fundamental Exact Sequence (3.10.1) associated to $X_n \xrightarrow{f_n} Y_n \hookrightarrow \mathfrak{Y}$, we deduce that $\widehat{\Omega}^1_{X_n/\mathfrak{Y}} = \Omega^1_{X_n/\mathfrak{Y}} = \Omega^1_{X_n/Y_n}$.

3.12. Given $\mathfrak{X}' \overset{i}{\hookrightarrow} \mathfrak{X}$ a closed immersion in NFS we have that the morphism $i^\sharp : i^{-1}(\mathcal{O}_{\mathfrak{X}}) \to \mathcal{O}_{\mathfrak{X}'}$ is an epimorphism. If $\mathcal{K} := \ker(i^\sharp)$ we call $\mathcal{C}_{\mathfrak{X}'/\mathfrak{X}} := \mathcal{K}/\mathcal{K}^2$ *the conormal sheaf of $\mathfrak{X}'$ in $\mathfrak{X}$.*

It is easily shown that $\mathcal{C}_{\mathfrak{X}'/\mathfrak{X}}$ satisfies the following properties:

(1) It is a coherent $\mathcal{O}_{\mathfrak{X}'}$-module.
(2) If $\mathfrak{X}' \subset \mathfrak{X}$ is a closed subscheme given by a coherent Ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$, then $\mathcal{C}_{\mathfrak{X}'/\mathfrak{X}} = i^*(\mathcal{I}/\mathcal{I}^2)$.

Proposition 3.13. *(Second Fundamental Exact Sequence) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo finite type morphism in* NFS*, and $\mathfrak{X}' \overset{i}{\hookrightarrow} \mathfrak{X}$ a closed immersion. There exists an exact sequence of coherent $\mathcal{O}_{\mathfrak{X}'}$-Modules*

$$\mathcal{C}_{\mathfrak{X}'/\mathfrak{X}} \xrightarrow{\delta} i^*\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \xrightarrow{\Phi} \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}} \to 0 \tag{3.13.1}$$

Proof. Morphism Φ is the differential of i and is defined by $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}a \otimes 1 \rightsquigarrow \widehat{d}_{\mathfrak{X}'/\mathfrak{Y}}i(a)$ (Proposition 3.7). If $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ is the Ideal that defines the closed subscheme $i(\mathfrak{X}') \subset \mathfrak{X}$ the morphism δ is the one induced by $\widehat{d}_{\mathfrak{X}/\mathfrak{Y}}|_{\mathcal{I}} : \mathcal{I} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$. Again the exactness is consequence of [EGA IV$_1$, (**0**, 20.7.20)]). □

As it happens in Sch the Second Fundamental Exact Sequence leads to a local description of the module of differentials of a pseudo finite type morphism between locally noetherian formal schemes.

3.14. Let $f : \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{Y} = \mathrm{Spf}(B)$ be a morphism in $\mathsf{NFS}_{\mathsf{af}}$ of pseudo finite type, then it factors as (see Proposition 1.7)

$$\mathfrak{X} = \mathrm{Spf}(A) \overset{j}{\hookrightarrow} \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}} = \mathrm{Spf}(B\{\mathbf{T}\}[[\mathbf{Z}]]) \xrightarrow{p} \mathfrak{Y} = \mathrm{Spf}(B)$$

where r, $s \in \mathbb{N}$, $\mathbf{T} = T_1, T_2, \ldots, T_r$ and $\mathbf{Z} = Z_1, Z_2, \ldots, Z_r$ two sets of indeterminates, p is the canonical projection and j is a closed immersion given by an Ideal $\mathcal{I} = I^{\triangle} \subset \mathcal{O}_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}}}$. Consider also a system of generators $I = \langle P_1, \ldots, P_k \rangle \subset B\{\mathbf{T}\}[[\mathbf{Z}]]$.

The Second Fundamental Exact Sequence (3.13.1) associated to $\mathfrak{X} \overset{j}{\hookrightarrow} \mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}} \xrightarrow{p} \mathfrak{Y}$ corresponds through the equivalence between the category of finite type A-modules and $\mathsf{Coh}(\mathfrak{X})$ 1.12.3, to the sequence

$$\frac{I}{I^2} \xrightarrow{\delta} \widehat{\Omega}^1_{B\{\mathbf{T}\}[[\mathbf{Z}]]/B} \otimes_{B\{\mathbf{T}\}[[\mathbf{Z}]]} A \xrightarrow{\Phi} \widehat{\Omega}^1_{A/B} \to 0. \tag{3.14.1}$$

Let us use the following abbreviation $\widehat{d} = \widehat{d}_{B\{\mathbf{T}\}[[\mathbf{Z}]]/B}$. Since

$$\widehat{\Omega}^1_{B\{\mathbf{T}\}[[\mathbf{Z}]]/B} \cong \widehat{\Omega}^1_{B[\mathbf{T},\mathbf{Z}]/B} \cong \bigoplus_{i=1}^{r} B\{\mathbf{T}\}[[\mathbf{Z}]]\widehat{d}T_i \oplus \bigoplus_{j=1}^{s} B\{\mathbf{T}\}[[\mathbf{Z}]]\widehat{d}Z_j$$

then $\{\widehat{d}T_1, \widehat{d}T_2, \ldots, \widehat{d}T_r, \widehat{d}Z_1, \ldots, \widehat{d}Z_s\}$ is a basis of the free $B\{\mathbf{T}\}[[\mathbf{Z}]]$-module $\widehat{\Omega}^1_{B\{\mathbf{T}\}[[\mathbf{Z}]]/B}$. Therefore, if $a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_{r+s}$ are the images of $T_1, T_2, \ldots, T_r, Z_1, \ldots, Z_s$ in A, by the definition of Φ we have

$$\widehat{\Omega}^1_{A/B} = \langle \widehat{d}_{A/B}a_1, \widehat{d}_{A/B}a_2, \ldots, \widehat{d}_{A/B}a_r, \widehat{d}_{A/B}a_{r+1}, \ldots, \widehat{d}_{A/B}a_{r+s} \rangle$$

and from the exactness of (3.14.1) it holds that

$$\widehat{\Omega}^1_{A/B} \cong \frac{\widehat{\Omega}^1_{B\{\mathbf{T}\}[[\mathbf{Z}]]/B} \otimes_{B\{\mathbf{T}\}[[\mathbf{Z}]]} A}{\langle \widehat{d}P_1 \otimes 1, \dots, \widehat{d}P_k \otimes 1 \rangle}$$

or, equivalently, since the functor $(-)^{\triangle}$ is exact on $\mathsf{Coh}(\mathfrak{X})$,

$$\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \cong \frac{\widehat{\Omega}^1_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}}/\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathbb{D}^s_{\mathbb{A}^r_{\mathfrak{Y}}}}} \mathcal{O}_{\mathfrak{X}}}{\langle \widehat{d}P_1 \otimes 1, \dots, \widehat{d}P_k \otimes 1 \rangle^{\triangle}}.$$

4. Differentials and infinitesimal lifting properties

Next we study some characterizations for a smooth, unramified and étale morphism between locally noetherian formal schemes. Above all, we will focus on the properties related to the module of differentials $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$. We highlight the importance of the Jacobian Criterion for affine formal schemes (Corollary 4.15) which allows us to determine when a closed formal subscheme of a smooth formal scheme is smooth rendering Zariski's Jacobian Criterion for topological rings (*cf.* [EGA IV$_1$, (**0**, 22.6.1)] into the present context.

Proposition 4.1. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ a morphism in* NFS*.*

(1) *Given $\{\mathfrak{U}_\alpha\}_{\alpha \in L}$ an open covering of $\mathfrak{X}$, f is smooth (unramified or étale) if, and only if, for all $\alpha \in L$, $f|_{\mathfrak{U}_\alpha} : \mathfrak{U}_\alpha \to \mathfrak{Y}$ is smooth (unramified or étale, respectively).*
(2) *If $\{\mathfrak{V}_\lambda\}_{\lambda \in J}$ is an open covering of $\mathfrak{Y}$, f is smooth (unramified or étale) if, and only if, for all $\lambda \in J$, $f|_{f^{-1}(\mathfrak{V}_\lambda)} : f^{-1}(\mathfrak{V}_\lambda) \to \mathfrak{V}_\lambda$ is smooth (unramified or étale, respectively).*

Proof. This may be proved similarly as the case of usual schemes [EGA IV$_4$, (17.1.6)] having in mind Propositions 2.9 and 2.13. □

Corollary 4.2. *The results 2.13, 2.14 and 4.1 are true if we replace the infinitesimal lifting properties by their adic counterparts.*

Proof. It is straightforward from this results in view of [EGA I, (10.12.1)] and Proposition 1.8. □

Remark. It follows from the two last results that in the local study of the infinitesimal lifting properties (with or without the adic hypothesis) over locally noetherian formal schemes, we can restrict to $\mathsf{NFS}_{\mathsf{af}}$.

4.3. We say that $f : \mathfrak{X} \to \mathfrak{Y}$ in NFS is *smooth (unramified or étale) at $x \in \mathfrak{X}$* if there exists an open subset $\mathfrak{U} \subset \mathfrak{X}$ with $x \in \mathfrak{U}$ such that $f|_{\mathfrak{U}}$ is smooth (unramified or étale, respectively).

By Proposition 4.1 it holds that f is smooth (unramified or étale) if, and only if, f is smooth (unramified or étale, respectively) at $x \in \mathfrak{X}$, $\forall x \in \mathfrak{X}$. Observe that the set of points $x \in \mathfrak{X}$ such that f is smooth (unramified, or étale) in x is an open subset of $\mathfrak{X}$.

In a forthcoming paper we will show how the infinitesimal lifting conditions in NFS at a given point depend only on the local rings.

Corollary 4.4. *Let X be in Sch and $X' \subset X$ a closed subscheme. Then the morphism of completion of X along X', $\kappa : X_{/X'} \to X$ is étale.*

Proof. Applying Proposition 4.1 we may assume that $X = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(A/I)$ are in $\mathsf{Sch}_{\mathsf{af}}$ with A a noetherian ring and $I \subset A$ an ideal. Then, it follows from Example 2.8.(5). □

Proposition 4.5. *Given $f : X \to Y$ in Sch, let $X' \subset X$ and $Y' \subset Y$ be closed subschemes such that $f(X') \subset Y'$.*

1. *If f is smooth (unramified or étale) then $\widehat{f} : X_{/X'} \to Y_{/Y'}$ is smooth (unramified or étale, respectively).*
2. *If moreover $X' = f^{-1}(Y')$ then $\widehat{f} : X_{/X'} \to Y_{/Y'}$ is adic smooth (adic unramified or adic étale, respectively).*

Proof. Let us consider the commutative diagram of locally noetherian formal schemes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \uparrow{\scriptstyle\kappa} & & \uparrow{\scriptstyle\kappa} \\ X_{/X'} & \xrightarrow{\widehat{f}} & Y_{/Y'} \end{array}$$

where the vertical arrows are morphisms of completion that, being slightly imprecise, we denote both by κ. Let us prove (1). If f is smooth (unramified or étale) by the last corollary and Proposition 2.9.(1) we have that $f \circ \kappa = \kappa \circ \widehat{f}$ is also smooth (unramified or étale). Since κ is étale from Proposition 2.13 we deduce that $\widehat{f}$ is smooth (unramified or étale, respectively). Assertion (2) is consequence of (1) and [EGA I, (10.13.6)]. □

Proposition 4.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism in NFS of pseudo finite type. The morphism f is unramified if, and only if, $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} = 0$.*

Proof. By Proposition 4.1 and Definition 3.1 we may suppose that $f : \mathfrak{X} \to \mathfrak{Y}$ is in $\mathsf{NFS}_{\mathsf{af}}$ and therefore the result follows from [EGA IV_1, (**0**, 20.7.4)]. □

Corollary 4.7. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be two pseudo finite type morphisms in NFS. Then f is unramified if, and only if, the morphism of $\mathcal{O}_{\mathfrak{X}}$-Modules $f^*(\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}}) \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$ is surjective.*

Proof. Use the last proposition and the First Fundamental Exact Sequence (3.10.1) associated to the morphisms $\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{S}$. □

Proposition 4.8. *(*cf. [Lipman, Nayak, Sastry 2005, Proposition 2.6.1]*) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism. Then f is flat and $\widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}}$ is a locally free $\mathcal{O}_{\mathfrak{X}}$-module of finite rank.*

Proof. Since it is a local question, we may assume that $f : \mathfrak{X} = \operatorname{Spf}(A) \to \mathfrak{Y} = \operatorname{Spf}(B)$ is in $\mathsf{NFS}_{\mathsf{af}}$ where $\phi \colon B \to A$ is a topological B-algebra that is formally smooth (see (2.2)). As for proving the flatness it suffices to show that for all maximal ideals $\mathfrak{p} \subset A$, $A_{\mathfrak{p}}$ is a flat $B_{\mathfrak{q}}$-module with $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Fix $\mathfrak{p} \subset A$ a prime ideal. By [EGA IV_1, (**0**, 19.3.5.(iv))] it holds that $A_{\mathfrak{p}}$ is a formally smooth $B_{\mathfrak{q}}$-algebra for the adic topologies and applying [EGA IV_1, (**0**, 19.3.8)] there results that $A_{\mathfrak{p}}$ is a formally smooth $B_{\mathfrak{q}}$-algebra for the topologies given by the maximal ideals. Then, by [EGA IV_1, (**0**, 19.7.1)] we have that $A_{\mathfrak{p}}$ is a flat $B_{\mathfrak{q}}$-module.

Let now $J \subset A$ be an ideal of definition of A. The A/J-module $\widehat{\Omega}^1_{A/B} \otimes_A A/J$ is projective. Indeed, given an exact sequence $L \xrightarrow{g} M \to 0$ of A/J-modules, the sequence

$$\operatorname{Hom}_{A/J}(\widehat{\Omega}^1_{A/B} \otimes_A A/J, L) \to \operatorname{Hom}_{A/J}(\widehat{\Omega}^1_{A/B} \otimes_A A/J, M) \to 0$$

is exact by the identity of functors on A/J-modules

$$\operatorname{Hom}_{A/J}(\widehat{\Omega}^1_{A/B} \otimes_A A/J, -) = \operatorname{Hom}_A(\widehat{\Omega}^1_{A/B}, -) = \operatorname{Der}_B(A, -)$$

and an argument analogous to the proof of [Matsumura 86, Theorem 28.5]. But $\widehat{\Omega}^1_{A/B}$ is a finite type A-module (see Proposition 3.3) by [EGA I, (**0**, 7.2.10)], then it follows that $\widehat{\Omega}^1_{A/B}$ is a projective A-module. The result is now a consequence of [EGA I, (10.10.8.6)]. □

Proposition 4.9. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a smooth morphism in* NFS*. For all pseudo finite type morphism $\mathfrak{Y} \to \mathfrak{S}$ in* NFS *the sequence of coherent $\mathcal{O}_{\mathfrak{X}}$-modules*

$$0 \to f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \xrightarrow{\Phi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \xrightarrow{\Psi} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0$$

defined in Proposition 3.10 is exact and locally split.

Proof. It is a local question, and follows from [Lipman, Nayak, Sastry 2005, Lemma 2.5.2] that is based on [EGA IV_1, (**0**, 20.7.17.3) and (**0**, 20.7.18)]. □

Corollary 4.10. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an étale morphism in* NFS*. For all pseudo finite type morphism $\mathfrak{Y} \to \mathfrak{S}$ in* NFS *it holds that*

$$f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \cong \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$$

Proof. It is a consequence of the last result and of Proposition 4.6. □

4.11. Given A a J-preadic ring, let $\widehat{A}$ be the completion of A for the J-adic topology and $A_n = A/J^{n+1}$, for all $n \in \mathbb{N}$. Take M'', M' and M A-modules, denote by $\widehat{M''}$, $\widehat{M'}$, $\widehat{M}$ their completions for the J-adic topology and let $\widehat{M''} \xrightarrow{u} \widehat{M'} \xrightarrow{v} \widehat{M}$ be a sequence of $\widehat{A}$-modules. It holds that

(1) If $0 \to \widehat{M''} \xrightarrow{u} \widehat{M'} \xrightarrow{v} \widehat{M} \to 0$ is a split exact sequence of $\widehat{A}$-modules then, for all $n \in \mathbb{N}$

$$0 \to M'' \otimes_A A_n \xrightarrow{u_n} M' \otimes_A A_n \xrightarrow{v_n} M \otimes_A A_n \to 0$$

is a split exact sequence.

(2) Reciprocally, if $M \otimes_A A_n$ is a projective A_n-module and
$$0 \to M'' \otimes_A A_n \xrightarrow{u_n} M' \otimes_A A_n \xrightarrow{v_n} M \otimes_A A_n \to 0$$
is a split exact sequence of A_n-modules, for all $n \in \mathbb{N}$, then
$$0 \to \widehat{M''} \to \widehat{M'} \to \widehat{M} \to 0 \tag{4.11.1}$$
is a split exact sequence of $\widehat{A}$-modules.

Assertion (1) is immediate. In order to prove (2), for all $n \in \mathbb{N}$ we have the following commutative diagrams:
$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'' \otimes_A A_{n+1} & \xrightarrow{u_{n+1}} & M' \otimes_A A_{n+1} & \xrightarrow{v_{n+1}} & M \otimes_A A_{n+1} & \to & 0 \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle h_n} & & \\
0 & \longrightarrow & M'' \otimes_A A_n & \xrightarrow{u_n} & M' \otimes_A A_n & \xrightarrow{v_n} & M \otimes_A A_n & \to & 0
\end{array}$$
where the rows are split exact sequences and the vertical maps are the canonical ones. Applying inverse limit we have that the sequence (4.11.1) is exact. Let us show that it splits. By hypothesis, for all $n \in \mathbb{N}$ there exists $t_n : M \otimes_A A_n \to M' \otimes_A A_n$ such that $v_n \circ t_n = 1$. From $\{t_n\}_{n\in\mathbb{N}}$ we are going to define a family of morphisms $\{t'_n : M \otimes_A A_n \to M' \otimes_A A_n\}_{n\in\mathbb{N}}$ such that
$$v_n \circ t'_n = 1 \qquad g_n \circ t'_{n+1} = t'_n \circ h_n \tag{4.11.2}$$
for all $n \in \mathbb{N}$. For $k = 0$ put $t'_0 := t_0$. Suppose that we have constructed t'_k verifying (4.11.2) for all $k \leq n$ and let us define t'_{n+1}. If $w_n := g_n \circ t_{n+1} - t'_n \circ h_n$ then $v_n \circ w_n = 0$ and therefore, $\operatorname{Im} w_n \subset \operatorname{Ker} v_n = \operatorname{Im} u_n$. Since $M \otimes_A A_{n+1}$ is a projective A_{n+1}-module, there exists $\theta_{n+1} : M \otimes_A A_{n+1} \to u_{n+1}(M'' \otimes_A A_{n+1})$ such that the following diagram is commutative
$$\begin{array}{ccc}
 & \nearrow^{\theta_{n+1}} & u_{n+1}(M'' \otimes_A A_{n+1}) \\
 & & \downarrow \\
M \otimes_A A_{n+1} & \xrightarrow{w_n} & u_n(M'' \otimes_A A_n).
\end{array}$$
If we put $t'_{n+1} := t_{n+1} - \theta_{n+1}$, it holds that $v_{n+1} \circ t'_{n+1} = 1$ and $g_n \circ t'_{n+1} = t'_n \circ h_n$. The morphism
$$t' := \varprojlim_{n\in\mathbb{N}} t'_n$$
satisfies that $v \circ t' = 1$ and the sequence (4.11.1) splits.

Proposition 4.12. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a pseudo finite type morphism in* NFS *and $g : \mathfrak{Y} \to \mathfrak{S}$ a smooth morphism in* NFS. *The following conditions are equivalent:*

(1) *f is smooth*
(2) *$g \circ f$ is smooth and the sequence*
$$0 \to f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}} \to \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \to 0$$
is exact and locally split.

Proof. The implication (1) $\Rightarrow$ (2) is consequence of Proposition 2.9 and of Proposition 4.9.

As for proving that (2) $\Rightarrow$ (1) we may suppose that $f = \mathrm{Spf}(\phi) \colon \mathfrak{X} = \mathrm{Spf}(A) \to \mathfrak{Y} = \mathrm{Spf}(B)$ and $g = \mathrm{Spf}(\psi) \colon \mathfrak{Y} = \mathrm{Spf}(B) \to \mathfrak{S} = \mathrm{Spf}(C)$ are in $\mathsf{NFS}_{\mathsf{af}}$ being B and A formally smooth C-algebras. Let us show that A is a formally smooth B-algebra. Let E be a discrete ring , $I \subset E$ a square zero ideal and consider the commutative diagram of continuous homomorphisms of topological rings

$$\begin{array}{ccccc} C & \xrightarrow{\psi} & B & \xrightarrow{\phi} & A \\ & & {\scriptstyle \lambda}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ & & E & \overset{j}{\twoheadrightarrow} & E/I. \end{array}$$

Since A is a formally smooth C-algebra, there exists a continuous homomorphism of topological C-algebras $v : A \to E$ such that $v \circ \phi \circ \psi = \lambda \circ \phi$ and $j \circ v = u$. Then by [EGA IV_1, (**0**, 20.1.1)]) we have that $d := \lambda - v \circ \phi \in \mathrm{Dercont}_C(B, E)$. From the hypothesis and considering the equivalence of categories 1.12.3 we have that the sequence of finite type A-modules

$$0 \to \widehat{\Omega}^1_{B/C} \otimes_B A \to \widehat{\Omega}^1_{A/C} \to \widehat{\Omega}^1_{A/B} \to 0$$

is exact and split. Besides, since the morphism v is continuous and E is discrete there exists $n \in \mathbb{N}$ such that E is an A/J^{n+1}-module. Therefore the induced map $\mathrm{Hom}_A(\widehat{\Omega}_{A/C}, E) \to \mathrm{Hom}_B(\widehat{\Omega}_{B/C}, E)$ is surjective and applying [EGA IV_1, (**0**, 20.4.8.2)] we have that the map

$$\mathrm{Der}_C(A, E) \to \mathrm{Der}_C(B, E)$$

is surjective too. It follows that there exists $d' \in \mathrm{Der}_C(A, E)$ such that $d' \circ \phi = d$. If we put $v' := v + d'$, we have that $v' \circ \phi = \lambda$ and $j \circ v' = u$. Therefore, A is a formally smooth B-algebra. $\square$

Corollary 4.13. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be two pseudo finite type morphisms in* NFS *such that $g \circ f$ and g are smooth. Then, f is étale if, and only if, $f^*\widehat{\Omega}^1_{\mathfrak{Y}/\mathfrak{S}} \cong \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{S}}$.*

Proof. Follows from the last proposition and Proposition 4.6. $\square$

Proposition 4.14. *(Zariski Jacobian criterion for preadic rings) Let $B \to A$ be a continuous morphism of preadic rings and suppose that A is a formally smooth B-algebra. Given an ideal $I \subset A$. Let us consider in $A' := A/I$ the topology induced by the topology of A. The following conditions are equivalent:*

1. *A' is a formally smooth B-algebra.*
2. *Given $J \subset A$ an ideal of definition of A, define $A'_n := A/(J^{n+1} + I)$. The sequence of A'_n-modules*

$$0 \to \frac{I}{I^2} \otimes_{A'} A'_n \xrightarrow{\delta_n} \Omega^1_{A/B} \otimes_A A'_n \xrightarrow{\Phi_n} \Omega^1_{A'/B} \otimes_{A'} A'_n \to 0$$

is exact and split, for all $n \in \mathbb{N}$.

(3) *The sequence of* $\widehat{A'}$*-modules*

$$0 \to \widehat{\frac{I}{I^2}} \xrightarrow{\delta} \Omega^1_{A/B} \widehat{\otimes}_A A' \xrightarrow{\Phi} \widehat{\Omega}^1_{A'/B} \to 0$$

is exact and split.

Proof. The fact that (1) ⇔ (2) follows from [EGA IV_1, (**0**, 22.6.1), (**0**, 19.1.5) and (**0**, 19.1.7)] and from the Second Fundamental Exact Sequence associated to the morphisms $B \to A \to A'$. Let us show that (2) ⇔ (3). Since A' is a formally smooth B-algebra, from [Matsumura 86, Theorem 28.5] we deduce that $\Omega^1_{A'/B} \otimes_{A'} A'_n$ is a projective A'_n-module, for all $n \in \mathbb{N}$ and, therefore, the result follows from 4.11.(2). □

Corollary 4.15. *(Zariski Jacobian criterion for formal schemes) Let* $f : \mathfrak{X} = \operatorname{Spf}(A) \to \mathfrak{Y} = \operatorname{Spf}(B)$ *be a smooth morphism in* $\mathsf{NFS}_{\mathsf{af}}$ *and* $\mathfrak{X}' \hookrightarrow \mathfrak{X}$ *a closed immersion given by an Ideal* $\mathcal{I} = I^{\triangle} \subset \mathcal{O}_{\mathfrak{X}}$. *The following conditions are equivalent:*

(1) *The composed morphism* $\mathfrak{X}' \hookrightarrow \mathfrak{X} \xrightarrow{f} \mathfrak{Y}$ *is smooth.*

(2) *Given* $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$ *an Ideal of definition, if* $\mathcal{O}_{X'_n} := \mathcal{O}_{\mathfrak{X}}/(\mathcal{J}^{n+1} + \mathcal{I})$, *the sequence of coherent* $\mathcal{O}_{X'_n}$*-Modules*

$$0 \to \frac{\mathcal{I}}{\mathcal{I}^2} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{O}_{X'_n} \xrightarrow{\delta_n} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X'_n} \xrightarrow{\Phi_n} \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}'}} \mathcal{O}_{X'_n} \to 0$$

is exact and locally split, for all $n \in \mathbb{N}$.

(3) *The sequence of coherent* $\mathcal{O}_{\mathfrak{X}'}$*-Modules*

$$0 \to \frac{\mathcal{I}}{\mathcal{I}^2} \xrightarrow{\delta} \widehat{\Omega}^1_{\mathfrak{X}/\mathfrak{Y}} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}'} \xrightarrow{\Phi} \widehat{\Omega}^1_{\mathfrak{X}'/\mathfrak{Y}} \to 0$$

is exact and locally split.

Proof. By Proposition 3.3 and the equivalence of categories (1.12.3) it is a consequence of the last proposition. □

Remark. The implication (1) ⇒ (3) is [Lipman, Nayak, Sastry 2005, Proposition 2.6.8], itself a generalization of [EGA IV_4, (17.2.5)].

References

[Alonso, Jeremías, Lipman 1999] Alonso Tarrío, L.; Jeremías Lopez, A.; Lipman, J.: Duality and flat base-change on formal schemes, pp. 3–90 in *Studies in duality on noetherian formal schemes and non-noetherian ordinary schemes*. Contemp. Math. **244**, Providence, RI: American Mathematical Society, 1999.

[Bourbaki 1989] Bourbaki, N.: *Commutative algebra, Chapters 1–7.* Elements of Mathematics. Springer-Verlag, Berlin-New York, 1989.

[EGA I] Grothendieck, A.; Dieudonné, J. A.: *Eléments of Géométrie Algébrique I*, Grundlehren der math. Wissenschaften **166**, Springer-Verlag, Heidelberg, 1971.

[EGA III_1] Grothendieck, A.; Dieudonné, J. A.: *Eléments of Géométrie Algébrique III, Étude cohomologique des faisceaux cohérents*, Publications Mathématiques, **8**, Institut des Hautes Études Scientifiques, Paris, 1961.

[EGA IV_1] Grothendieck, A.; Dieudonné, J. A.: *Eléments of Géométrie Algébrique IV, Étude locale des schémas et des morphismes of schémas (première partie)*, Publications Mathématiques, **20**, Institut des Hautes Études Scientifiques, Paris, 1964.

[EGA IV_4] Grothendieck, A.; Dieudonné, J. A.: *Eléments of Géométrie Algébrique IV, Étude locale des schémas et des morphismes of schémas (quatrième partie)*, Publications Mathématiques, **32**, Institut des Hautes Études Scientifiques, Paris, 1967.

[Hartshorne 75] Hartshorne, R.: *On the of Rham cohomology of algebraic varieties. Pub. Math. IHES*, **45** (1975), 5–99.

[Hartshorne 77] Hartshorne, R.: *Algebraic geometry*, Graduate Texts in Mathematics, No. **52**. Springer-Verlag, New York, 1977.

[Henrio 2000] Henrio, Y.: Disques et couronnes ultramétriques, pp. 21–32 in *Courbes semi-stables et groupe fondamental en géométrie algébrique (Luminy, 1998)*, Progr. Math., **187**, Birkhäuser, Basel, 2000.

[Lipman, Nayak, Sastry 2005] Lipman, J.; Nayak, S.; Sastry P.: Pseudofunctorial behavior of Cousin complexes on formal schemes, pp. 3–133 in *Variance and duality for Cousin complexes on formal schemes*. Contemp. Math., **375**, Amer. Math. Soc., Providence, RI, 2005.

[Matsumura 86] Matsumura, H.: *Commutative ring theory*, Cambridge University Press, Cambridge, 1986.

[Raynaud 74] Raynaud, M.: Géométrie analytique rigide d'après Tate, Kiehl,... *Mémoires de la Société Mathématique de France*, **39-40** (1974), 319–327.

[Strickland 99] Strickland, N. P.: Formal schemes and formal groups. *Homotopy invariant algebraic structures* (Baltimore, MD, 1998), 263–352, *Contemp. Math.*, **239**, Amer. Math. Soc., Providence, RI, 1999.

[Tennison 75] Tennison, B. R.: *Sheaf Theory*, London Mathematical Society Lecture Note Series, **20**, Cambridge University Press, Cambridge, 1975.

[Yekutieli 98] Yekutieli, A.: Smooth formal embeddings and the residue complex. *Canad. J. Math.* **50** (1998), no. 4, 863–896.

Departamento de Álxebra, Facultade de Matemáticas, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela, SPAIN
E-mail address: leoalonso@usc.es
URL: http://web.usc.es/~lalonso/

Departamento de Álxebra, Facultade de Matemáticas, Universidade de Santiago de Compostela, E-15782 Santiago de Compostela, SPAIN
E-mail address: jeremias@usc.es

Departamento de Matemáticas, Escola Superior de Enxeñería Informática, Campus de Ourense, Univ. de Vigo, E-32004 Ourense, Spain
E-mail address: martapr@uvigo.es